Paying Back a Loan

We illustrate with an example:

Example 1 *On June 1, Dina withdrew \$600 from her credit card account to buy a "good" used car. The annual interest rate charged by the credit card company is a whooping 24% compounded monthly. She decided to pay back \$100 per month beginning June 30. How long will it take her to pay off the loan and the interest, and what will her last payment be? The answer is NOT 6 months because she is charged interest at the end of every month.*

Since she pays the loan in a relatively short time, let us use ***brute force*** *to solve this problem. This means calculating what she owes as the months go by until the bill is paid off. We need the interest rate per month, and it is* $j = \frac{24}{1200} = 0.02$*. At the beginning of June, she owed \$600. At the end of the month, the interest earned on what she owed during the month must be added. Therefore, at the end of June she owed*

$$600 + 600 \times 0.02 = 600\,(1.02) \text{ dollars}$$

After she made a \$100 payment she owed $600\,(1.02) - 100 = 512$ *dollars. This is the amount we use to calculate interest at the end of July. Indeed at the end of July, after making the \$100 payment she owes* $512 + 512\,(0.02) - 100 = 422.24$ *dollars. Complete the table below*

Month	*June*	*July*	*August*	*September*	*October*	*November*	*December*
Debt at beginning of month	*\$600*	*\$512*	*\$422.24*				
Debt at end of month before making \$100 payment	*\$612*	*\$522.24*					
Debt at end of month after making \$100 payment	*\$512*	*\$422.24*					

If you cranked the numbers correctly, you should have concluded that at the end of December, before making a payment, she owed \$45.78 (or a number close to this). Therefore she paid off the loan in 7 months and her last payment was \$45.78.

To get a pattern that leads to a formula, we keep 1.02 *and its exponents. Thus the amount she owed at the end of June, after making a payment is kept as* $600\,(1.02) - 100$*. Two months after getting the loan she owed* $[600\,(1.02) - 100] +$ *interest on* $[600\,(1.02) - 100]$*, which adds up to*

$$\begin{aligned} [600\,(1.02) - 100] + \; [600\,(1.02) - 100]\,(0.02) &= 600\,(1.02) + 600\,(1.02)\,(0.02) - [100 + 100\,(0.02)] \\ &= 600\,(1.02)^2 - 100\,(1.02) \end{aligned}$$

After she makes the \$100 payment, she owed

$$600\,(1.02)^2 - 100\,(1.02) - 100 = 600\,(1.02)^2 - 100\,(1 + 1.02)$$

It is shown in a similar way that the amount, in dollars, she owed after three months is

$$600\,(1.02)^3 - 100\left(1 + 1.02 + 1.02^2\right)$$

The table below gives the amounts she owed as the months went by

End of	*The amount, in dollars, she owed just after making the $100 payment*
1st month	$600\,(1.02) - 100$
2nd month	$600\,(1.02)^2 - 100\,(1 + 1.02)$
3rd month	$600\,(1.02)^3 - 100\left(1 + 1.02 + 1.02^2\right)$
4th month	$600\,(1.02)^4 - 100\left(1 + 1.02 + 1.02^2 + 1.02^3\right)$
5th month	$600\,(1.02)^5 - 100\left(1 + 1.02 + 1.02^2 + 1.02^3 + 1.02^4\right)$
6th month	$600\,(1.02)^6 - 100\left(1 + 1.02 + 1.02^2 + 1.02^3 + 1.02^4 + 1.02^5\right)$

In practice, loans are repaid over long periods of time. For example, a car loan may take 5 years to pay off and a house loan may take over 20 years. In such cases, the brute force method we used in the first part of the above solution is unpractical. Moreover, when one gets a loan from a lending institution, one does not decide one's monthly payment. It is the institution that decides and, in most instances, it will require that the borrower makes **equal** payments until the loan is paid off. The brute force method above cannot be used to determine such payments. The following example generalizes the computations given in the second table of the above example.

Example 2 *Lakeshia wishes to borrow $5000 from a bank for a project. The interest rate on the loan is 12% compounded monthly. What will her monthly payment be if the loan must be paid off in 48 equal monthly payments? (Of course it is NOT* 5000 *divided by the total number of months in 4 years, because she has to pay interest.)*

The interest rate per month is $\left(\dfrac{12}{100}\right) \div 12 = 0.01$. *We do not know the monthly payment; let it be* P *dollars. As we did in Dina's case, we trace what she owes as the months go by, and keep the exponents of* 1.01*. To simplify the description of the solution, we assume that she will get the loan at the beginning of January and will be making payments at month-ends, beginning January 31.*

At the end of January, soon after making the first payment of P *dollars, she will be owing 5000 dollars plus the interest on 5000 dollars minus the payment of* P *dollars. This adds up to*

$$5000 + 5000 \times 0.01 - P = 5000\,(1 + 0.01) - P = 5000\,(1.01) - P \text{ dollars}$$

At the end of February, soon after making the second payment, she will be owing $5000\,(1.01) - P$ *dollars plus the interest on* $5000\,(1.01) - P$ *dollars minus the payment of* P *dollars which comes to*

$$5000\,(1.01) - P + [5000\,(1.01) - P] \times 0.01 - P = 5000\,(1.01)^2 - P(1 + 1.01) \text{ dollars}$$

At the end of March, soon after making the third payment, she will be owing

$$\begin{aligned} & 5000\,(1.01)^2 - P(1 + 1.01) + \left[5000\,(1.01)^2 - P(1 + 1.01)\right] \times 0.01 - P \\ = \; & 5000\,(1.01)^3 - P(1 + 1.01 + 1.01^2) \text{ dollars} \end{aligned}$$

Soon after making the 4th payment, she will be owing

$$5000\,(1.01)^4 - P(1 + 1.01 + 1.01^2 + 1.01^3) \text{ dollars}$$

In general, soon after making the nth payment she will be owing

$$5000\,(1.01)^n - P(1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{n-1}) \text{ dollars}$$

In particular, soon after making the 48th payment, she will be owing

$$5000\,(1.01)^{48} - P(1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{47}) \text{ dollars}$$

Since the loan must be paid off in 48 months, she will be owing zero dollars, therefore

$$5000\,(1.01)^{48} - P(1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{47}) = 0$$

This is the equation we have to solve for P. *As expected, we use the formula* $1+x+x^2+\cdots+x^n = \dfrac{x^{n+1}-1}{x-1}$ *with* $x = 1.01$ *and* $n = 47$ *to get*

$$5000\,(1.01)^{48} = P\left(\frac{1.01^{48}-1}{0.01}\right)$$

A calculator gives $5000\,(1.01)^{48} = 8061.130388$ *and* $\dfrac{1.01^{48}-1}{0.01} = 61.22260777$, *therefore*

$$8061.130388 = P\,(61.22260777)$$

Solving the above equation gives $P = 8061.130388 \div 61.22260777 = 131.67$ *dollars , to the nearest cent. In other words, her monthly payments will be* \$131.67

Note that the 48 payments she makes add up to $131.67 \times 48 = 6320.16$ *dollars. She borrowed* \$5000. *The difference* $6320.16 - 5000 = 1320.16$ *dollars is the total amount of interest she paid. It is pretty high because of the high 12% interest rate.*

When we examine the equation $5000\,(1.01)^{48} = P(1 + 1.01 + 1.01^2 + \cdots + 1.0^{46} + P1.01^{47})$ *that has to be solved to get the monthly payment, we recognize* $5000\,(1.01)^{48}$ *as the 48-month future value of the \$5000 loan. The right hand side* $P(1 + 1.01 + 1.01^2 + \cdots + 1.0^{46} + P1.01^{47})$ *is the future value of the annuity consisting of the 48 monthly payments of* P *dollars each. Therefore* ***the future value of the loan equals the future value of the monthly payments****. This is one of the keys to solving these kinds of problems.*

Generalizing the above result:

Imagine securing a loan of A dollars under the following conditions: A total of n equal periods are specified and you will be required to make the same payment at the end of each period, starting with the first period. (The period may be a month, in which case you will make monthly payments, or a year or a week). The loan will be made available at the beginning of the first period and interest will be calculated beginning the first day of the first period. The interest rate per period will be j and it will be fixed for the life of the loan. What will be your payment per period?

To answer the question, we calculate the future value of the loan on the last day on the last period. Since the interest rate per period is j the required future value of the loan is

$$A\,(1+j)^n$$

We do not know the payment per period. Let it be P dollars. The n payments of P dollars each form an annuity with future value

$$P\left[\frac{(1+j)^n - 1}{j}\right]$$

To determine P, we use the fact that **the future value of the loan should equal the future value of the above annuity**. This gives the equation

$$P\left[\frac{(1+j)^n - 1}{j}\right] = A\,(1+j)^n$$

which we have to solve for P. Since A and j are known, it is a matter of using a calculator to determine $\dfrac{(1+j)^n - 1}{j}$ and $A\,(1+j)^n$ then solve for P.

Exercise 3

1. *Byron would like to buy a home musical system for \$1500. He can get a loan from a lending institution that charges interest at the rate of 18% per year, compounded monthly. You are required to figure out what his monthly payments will be if he pays off the loan in 24 equal payments, (i.e. in 2 years). Assume that she gets the loan on January 1 and starts making monthly payments on January 31.*

   (a) *What is the interest rate per month?*

   (b) *What is the 24-month future value of the \$1500 loan?*

   (c) *Say his monthly payments are P dollars, paid every end of month for 24 months, what is the future value of the annuity consisting of the 24 payments?*

   (d) *Form an appropriate equation and solve for P.*

   (e) *What will be the total amount of interest he will have to pay?*

   (f) *What will be the monthly payments and the total amount of interest if he opts to pay off the loan in one year?*

2. *James needs \$225,000 to buy a house. A lending institution has agreed to lend him the money under the following conditions:*

   (a) *He must pay off the loan in equal monthly payments made at the end of every month.*
   (b) *He must pay off the loan in 30 or less years.*
   (c) *He must pay interest at the rate of 6.9% per year, compounded monthly.*

   He wants you to help him figure out his monthly payments if he opts to pay off the loan in 30 years, (which comes to 360 equal monthly payments).

   Determine the interest rate per month.

   What is the 360-month future value of the \$225,000 loan?

   Let the monthly payment be P dollars. What is the future value of the annuity consisting of the 360 monthly payments?

   Form an equation for P and solve it.

   (a) *What is the total amount of interest he pays over the 30 years?*

   (b) *What would be the monthly payment if he opts for a 15 year loan at the same interest rate?*

   (c) *What is the total amount of interest he would pay if he opts for a 15 year loan?*

   (d) *Suppose the interest rate is reduced to 3% per year, compounded monthly.*

      1. *What would be the monthly payment and the total amount of interest he would pay if he opts to pay off the loan in 30 years?*
      2. *What would be the monthly payment and the total amount of interest he would pay if he opts to pay off the loan in 15 years?*

3. *Aisha is looking for a \$15,000 loan to buy a car. One financial institution has agreed to give her the loan under the following conditions:*

   (a) *She must pay interest at the rate of 15% per year, compounded monthly.*
   (b) *She must pay off the loan in equal monthly payments made at month-ends.*
   (c) *She must pay off the loan in 5 or less years.*

   (a) *What will her monthly payments be if she opts to pay off the loan in 5 years and how much interest will she pay?*

   (b) *How much less interest will she pay if she chooses to pay off the loan in 3 years?*

4. *To buy a house, Ms. Morgan is required to make a down payment equal to 20% of the value of the house then borrow the balance from a bank that charges interest at an annual rate of 8.4% compounded monthly. The bank loan has to be paid off with equal monthly payments, (paid at the end of each month), in 25 years. Show that if she chooses to buy a $250000 house then her monthly payments will be $1597 to the nearest cent. Assume that she would get the loan at the beginning of a month and start making monthly payments on the last day of each month until the loan is paid off.*

5. *This exercise refers to Ms. Morgan's payments in Exercise 4 above. A consumer advocate suggested to her that instead of making monthly payments of $1597 each, she should consider dividing 1597 into two and pay $798.50 every half month. In that way she would pay off the loan a lot faster. Assuming that she gets the loan on January 1, here is one way of figuring out whether it makes a substantial difference:*

   (a) *The interest rate per half month is* $8.4\% \div 24 = 0.0035$. *After making the 20% down payment, she owes $200000 on January 1. She makes a $798.50 payment on January 15. Show that after making the payment, she owes*

   $$200000\,(1.0035) - 798.50 \text{ dollars}$$

   (b) *Show that after making the $798.50 payment on January 31, she owes*

   $$200000\,(1.0035)^2 - 798.50\,(1 + 1.0035) \text{ dollars}$$

   (c) *Show that after making the third $798.50 payment in mid-February, she owes*

   $$200000\,(1.0035)^3 - 798.50(1 + 1.0035 + 1.0035^2) \text{ dollars}$$

   (d) *Show that after making a total of k payments of $798.50 each, she owes*

   $$200000\,(1.0035)^k - 798.50(1 + 1.0035 + 1.0035^2 + \cdots + 1.0035^{k-1}) \text{ dollars}$$

   (e) *You need k such that* $200000\,(1.0035)^k - 798.50(1 + 1.0035 + 1.0035^2 + \cdots + 1.0035^{k-1}) = 0$. *Use the formula* $1 + x + x^2 + \cdots + x^{k-1} = \dfrac{x^k - 1}{x - 1}$ *to simplify* $1 + 1.0035 + 1.0035^2 + \cdots + 1.0035^{k-1}$ *then show that k is approximately 598.95.*

   (f) *This means that she will have to make 598.95 payments of 798.50 dollars each, instead of 300 payments of 1597 dollars each. How much does she save by doing precisely that?*

   (g) *What is her monthly saving?*

6. *Oscar goes to college and he chose to rent an apartment off campus. The rent per month is $450. On January 1, his sponsors deposited $5400, (for the sole purpose of paying Oscar's rent), into an account that pays interest at the rate of 6% per year, compounded monthly. Starting January 31 of the same year, Oscar was allowed to withdraw $450 at each month-end to pay his rent. How much money remained in the account on December 31 of the same year after he had withdrawn the December rent? The answer is NOT* 0 *because the money in the account earns interest.*

   Answer the same question if the sponsors deposited $90000 instead of $5400.