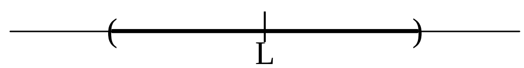
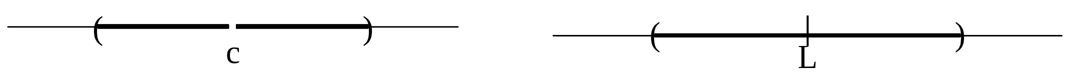
A Precise Definition of a Limit

Let f be a given function and c be a given real number. We have so far defined the limit of f as x approaches c intuitively: it is the number that is close to all the values $f(x)$ of f when x is close to c. Unfortunately, this definition cannot be used to prove general statements about limits. For example, we cannot use it to prove that if f has limit l and g has limit m as x approaches a number c then $f+g$ has limit $l+m$ as x approaches c. The most we can do with it, (which is not a rigorous proof), is to take specific functions f and g which have limits l and m respectively as x approaches a specific point c and verify that the sum $f(x)+g(x)$ is close to $l+m$ when x is close to c.

In this section we derive a precise definition of a limit that can be used in proofs. We get it by replacing the vague phrase "*when x is close to c then $f(x)$ is close to L*" with a more precise statement. To this end, note that if you are asked to shade the numbers *on the number line* that you consider to be close to a given number L, you will, most probably, shade a small interval I centered at L. It may be denoted by $(L-\varepsilon, L+\varepsilon)$ where ε is some positive number. (The size of ε will depend on your choice of precision.)



Having shaded such an interval, you would say that a number y is close to L if y is in the shaded region. In particular, the values $f(x)$ of a given function f should be close to L if, (when you plot them on the number line), they fall in the shaded interval. Therefore, to convince you that numbers x close to c give values $f(x)$ close to L, it suffices to produce an interval $(c-\delta, c+\delta)$ centered at c such that every number x in $(c-\delta, c+\delta)$ gives a value $f(x)$ in your interval $(L-\varepsilon, L+\varepsilon)$. Actually, we have to exclude the point c itself because it may not be in the domain of f. Therefore we have to ensure that every number x in $(c-\delta, c)\cup(c, c+\delta)$ gives a value $f(x)$ in $(L-\varepsilon, L+\varepsilon)$. The set $(c-\delta, c)\cup(c, c+\delta)$ is called a **punctured interval** centered at c.



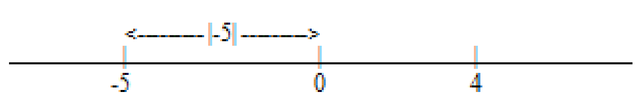
If x is in the shaded region centred at c then $f(x)$ is in the shaded region centred at L.

If we are to convince every one who may show up, we must be in a position to carry out the above steps for **any** positive number ε. This suggests the following definition:

Definition 1 *A function $f(x)$ has limit L as x approaches a number c if, given* ***any interval*** *$(L-\varepsilon, L+\varepsilon)$, we can find a punctured interval $(c-\delta, c)\cup(c, c+\delta)$ such that $f(x)\in(L-\varepsilon, L+\varepsilon)$ for all x in $(c-\delta, c)\cup(c, c+\delta)$. We then write $\lim\limits_{x\to c} f(x) = L$.*

Admittedly, this is not the definition you are likely to find in general use. The standard one is given in terms of absolute values. To make sure you are on board, we briefly review them here.

The absolute value of a given number x, (denoted by $|x|$), is the distance from the point representing x on the number line, to the point that represents 0, which we call the origin. For example, $|-5| = 5$ and $|4| = 4$. The figure below shows the distance, indicated by the dotted line, from the point representing -5 to the origin.



Clearly, if x is non-negative then $|x| = x$ and if x is negative then $|x|$ is obtained by simply changing the sign of x. Thus

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since a negative number is less than any non-negative number, it follows that $x < |x|$ if x is negative. Combining this with the fact that $x = |x|$ if x is non-negative, we conclude that

$$x \leq |x| \text{ for all real numbers } x$$

An alternative definition of the absolute value $|x|$ of a real number x that makes no mention of distances is the following:

$$|x| = \sqrt{(x)^2} \tag{1}$$

where $\sqrt{(x)^2}$ denotes the positive square root of $(x)^2$. For example,

$$\sqrt{(-5)^2} = \sqrt{25} = 5$$

confirming what we got above. It also follows from (1) that $|x|^2 = x^2$ for any real number x. Using (1), we find that if x and y are any real numbers then

$$\begin{aligned} |x+y|^2 &= \left(\sqrt{(x+y)^2}\right)^2 = (x+y)^2 = x^2 + 2xy + y^2 \\ &= |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

In other words, if x and y are any real numbers then $|x+y|^2 \leq (|x| + |y|)^2$. When we take square roots of both sides, we get

$$|x+y| \leq |x| + |y| \tag{2}$$

This is called the triangle inequality and we use this inequality repeatedly to prove a number of statements.

There is nothing special about the number 0. We can consider the distance from the point representing a number x to the point representing a number y on the number line. We denote it by $|x-y|$ or $|y-x|$ because the order is immaterial. For example, $|4-7|$ is the distance from the point representing 4 to the point representing 7, which is 3; and $|5-(-6)|$ is the distance from the point representing 5 to the point representing -6, which is 11.

Turning to definition 1, we note that an interval $(l-\varepsilon, l+\varepsilon)$ centered at l consists of all the numbers y whose distance from l is less than ε. Therefore $f(x) \in (l-\varepsilon, l+\varepsilon)$ if $|f(x) - l| < \varepsilon$. Likewise the interval $(c-\delta, c+\delta)$ consists of all the numbers x whose distance from c is less than δ. When we leave out the center c of the interval, we get the numbers x whose distance from c is strictly bigger than 0 and is less than δ. Therefore, a number x is in the punctured interval $(c-\delta, c) \cup (c, c+\delta)$ if $0 < |x-c| < \delta$. Now definition 1 may be given in the following equivalent form:

Definition 2 *A function $f(x)$ has limit L as x approaches a number c if, given any positive number ε, we can find a positive number δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |c-x| < \delta$.*

Example 3 *We show, using definition 2, that $f(x) = 3x+5$ has limit 11 as x approaches 2. To this end let ε be any positive number. We have to find a number $\delta > 0$ such that $|f(x) - 11| < \varepsilon$ whenever $0 < |x-2| < \delta$. The first step is to simplify the expression $|f(x) - 11|$:*

$$|f(x) - 11| = |3x + 5 - 11| = |3x - 6| = 3|x-2|$$

Therefore we have to produce a number $\delta > 0$ such that $3|x-2| < \varepsilon$ whenever $0 < |x-2| < \delta$. Clearly $3|x-2| < \varepsilon$ whenever $0 < |x-2| < \frac{1}{3}\varepsilon$, therefore any $\delta \leq \frac{\varepsilon}{3}$ will do.

Example 4 *Let* $g(x) = \dfrac{x-1}{x^3-1}$, $x \neq 1$. *Then* g *has limit* $\frac{1}{3}$ *as* x *approaches 1. To prove it using the precise definition of a limit, let* ε *be any positive number. We have to find a number* $\delta > 0$ *such that* $\left|g(x) - \frac{1}{3}\right| < \varepsilon$ *whenever* $0 < |x-2| < \delta$. *A number* x *close to 1 may be written in the form* $x = 1+h$ *where* h *is close to 0. This change of variable gives expressions that are easier to factor. Indeed*

$$\begin{aligned}\left|g(x) - \frac{1}{3}\right| &= \left|\frac{x-1}{x^3-1} - \frac{1}{3}\right| = \left|\frac{1+h-1}{(1+h)^3-1} - \frac{1}{3}\right| = \left|\frac{h}{h^3+3h^2+3h} - \frac{1}{3}\right| \\ &= \left|\frac{1}{h^2+3h+3} - \frac{1}{3}\right| = \left|\frac{3-h^2-3h-3}{3\left(h^2+3h+3\right)}\right| = \frac{|h|\,|h+3|}{3\left|h^2+3h+3\right|}\end{aligned}$$

We have to determine the numbers h *such that*

$$\frac{|h|\,|h+3|}{3\left|h^2+3h+3\right|} < \varepsilon \tag{3}$$

This is a fairly difficult inequality to solve. We replace it with a simpler one as follows: Since it is the numbers x *close to* 1 *that are relevant to the limit of* g *as* x *approaches* 1, *we may restrict ourselves to numbers* x *in a suitable interval centered at 1. An example is* $(0,2)$. *(Of course there is nothing special about this choice. Any other convenient one will also do.) Then we can put bounds on the terms* $|h+3|$ *and* $\left|h^2+3h+3\right|$. *Indeed, if* $x = 1+h$ *is in* $(0,2)$ *then* $-1 < h < 1$, *which implies that*

$$|h+3| < 4 \text{ and } \left|h^2+3h+3\right| > 1.$$

(We need to replace $|h+3| < 4$ *with a bigger number because it is in the numerator of (3). On the other hand,* $\left|h^2+3h+3\right|$ *is in the denominator, therefore we should replace it with a smaller number.) It now follows that if* x *is in* $(0,2)$ *then*

$$\left|g(x) - \frac{1}{3}\right| < \frac{|h| \cdot 4}{3 \cdot 1} = \frac{4\,|h|}{3} = \frac{4\,|x-1|}{3}.$$

Recall that we need a $\delta > 0$ *such that* $\left|g(x) - \frac{1}{3}\right| < \varepsilon$ *when* $|x-1| < \delta$. *Since*

$$\left|g(x) - \frac{1}{3}\right| < \frac{4\,|x-1|}{3}$$

for all numbers x *in* $(0,2)$, *it suffices to find a* $\delta > 0$ *such that*

$$\frac{4\,|x-1|}{3} < \varepsilon$$

when $|x-1| < \delta$. *This is a fairly easy inequality to solve. Any* $\delta < \frac{3\varepsilon}{4}$ *will do. (In addition, the* δ *should be smaller than 1 because* x *must be in* $(0,2)$*).*

Example 5 *Let* $g(x) = x$ *and* c *be any real number. Then* $\lim_{x\to c} g(x) = c$. *To prove it, let* $\varepsilon > 0$ *be given. We have to find a number* $\delta > 0$ *such that* $|g(x) - c| < \varepsilon$ *whenever* $0 < |x-c| < \delta$. *Since* $|g(x) - c| = |x - c|$, *the choice* $\delta = \varepsilon$ *will do. This proves that* $\lim_{x\to c} g(x) = c$.

Example 6 *Let* $g(x) = k$, *where* k *is a constant, and* c *be any real number. Then* $\lim_{x\to c} g(x) = k$. *To prove it, let* $\varepsilon > 0$ *be given. We have to find a number* $\delta > 0$ *such that* $|g(x) - k| < \varepsilon$ *whenever* $0 < |x-c| < \delta$. *But the inequality* $|f(x) - k| < \varepsilon$ *is satisfied by any* x *you choose. Therefore any positive number* δ *will do.*

Example 7 *Let* $g(x) = \sqrt{x}$, $x > 0$, *and* c *be any positive number. Then* $\lim_{x\to c} g(x) = \sqrt{c}$. *For a proof, let* $\varepsilon > 0$ *be given. We have to find a* $\delta > 0$ *such that* $|g(x) - \sqrt{c}| < \varepsilon$ *whenever* $0 < |x-c| < \delta$. *Consider the expression*

$$|g(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}|$$

To go beyond this, we have to rationalize the numerator of $|\sqrt{x}-\sqrt{c}| = \dfrac{|\sqrt{x}-\sqrt{c}|}{1}$. *The result is*

$$\left|\sqrt{x}-\sqrt{c}\right| = \frac{|\sqrt{x}-\sqrt{c}|\,|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|} = \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|}$$

Note that $\dfrac{|x-c|}{|\sqrt{x}+\sqrt{c}|} < \dfrac{|x-c|}{\sqrt{c}}$, *(because, in* $\dfrac{|x-c|}{\sqrt{c}}$, *we are dividing by a smaller number). Therefore*

$$\left|\sqrt{x}-\sqrt{c}\right| = \frac{|\sqrt{x}-\sqrt{c}|\,|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|} = \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} < \frac{|x-c|}{\sqrt{c}} \tag{4}$$

We are looking for a $\delta > 0$ *such that* $|\sqrt{x}-\sqrt{c}|$ *whenever* $0 < |x-c| < \delta$. *It suffices to find a* $\delta > 0$ *such that* $\dfrac{|x-c|}{\sqrt{c}} < \varepsilon$ *whenever* $0 < |x-c| < \delta$. *Clearly, any* $\delta < (\sqrt{c})\,\varepsilon$ *will do.*