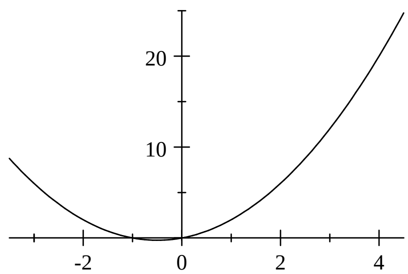
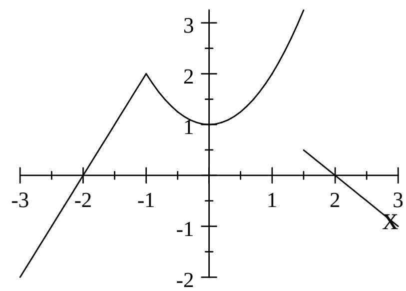
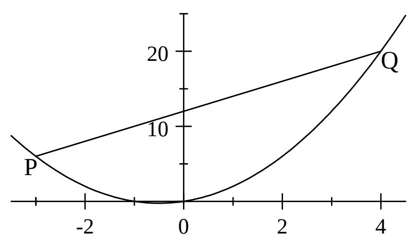
The Mean Value Theorem and Some Applications

The Mean Value Theorem is a statement about chords and tangents drawn on **smooth** graphs. Intuitively, a function f has a **smooth** graph on an interval $[a, b]$ if we can draw a tangent at every point $(x, f(x))$, $x \in [a, b]$, on its graph The left figure below is the graph of $f(x) = x^2 + x$ on the interval $[-3.5, 4.5]$, (arbitrarily chosen). Its graph is smooth since we can draw a tangent at any point $(x, f(x))$. The graph in the second figure is not smooth on the interval $[-3, 3]$ because we cannot draw a tangent at $(-1, 2)$ or $(1.5, 3.4)$.

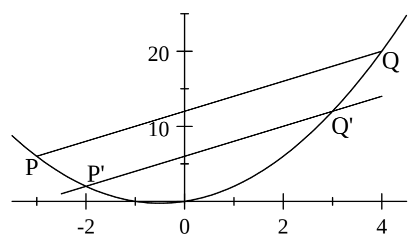




Let f have a smooth graph on an interval $[a, b]$. Let c and d be points such that $a \le c < d \le b$. Consider the chord PQ joining $P(c, f(c))$ and $Q(d, f(d))$ on the graph of f, (we used $f(x) = x^2 + x$, $c = -2.5$ and $d = 4$ in the figure below). Let $P'Q'$ be a variable line segment that is always parallel to PQ.

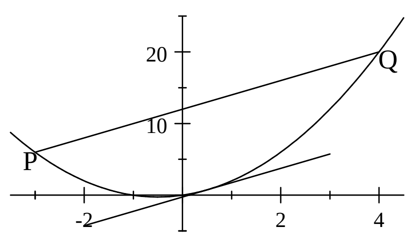


A chord PQ joining $(-3, 6)$ and $(4, 20)$



PQ and $P'Q'$

The mean value theorem asserts that if you slide $P'Q'$ far enough in the right direction, it becomes a tangent to the graph of f at some point $(\theta, f(\theta))$ where θ is between c and d.



It becomes a tangent

Since the slope of PQ is $\dfrac{f(d) - f(c)}{d - c}$, the theorem asserts that there is a number θ between c and d such that

$$f'(\theta) = \frac{f(d) - f(c)}{d - c} \tag{1}$$

This is conveniently written as $f(d) - f(c) = (d - c)\, f'(\theta)$.

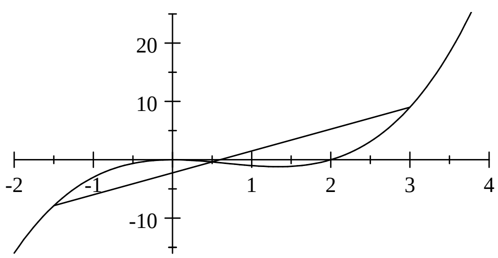
In the case of $f(x) = x^2 + x$ and points $c = -2.5$, $d = 4$, it turns out that θ is approximately equal to 0.73. We obtained it is by solving the equation

$$2\theta + 1 = f'(\theta) = \frac{f(4) - f(-2.5)}{4 - (-2.5)} = \frac{20 - 8.75}{6.5} = \frac{11.25}{6.5}$$

The theorem is generally stated in the following form:

Theorem 1 ***(The Mean Value Theorem)*** *Let f have a smooth graph on an interval $[a, b]$ and c, d be points in $[a, b]$ with $c < d$. Then there is a number θ between c and d such that $f(d) - f(c) = (d - c)\, f'(\theta)$.*

Note that the theorem does not claim exactly one point θ satisfying the above conditions. It asserts that there is at least one point; leaving open the possibility of two or more. For example, let $f(x) = x^3 - 2x^2$, and choose $c = -1.5$ and $d = 3$.



The theorem states that there is a number θ between -1.5 and 3 such that

$$f(3) - f(-1.5) = f'(\theta)\,(3 - (-1.5)) = 4.5 f'(\theta).$$

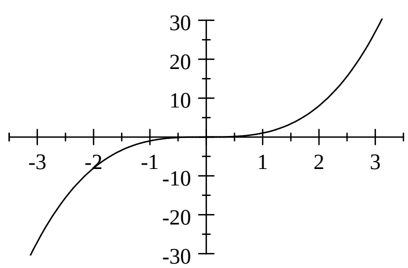
Since $f(3) - f(-1.5) = 9 - (-7.875) = 16.875$ and $f'(x) = 3x^2 - 4x$, θ satisfies the quadratic equation

$$3\left(\theta^2\right) - 4\theta = \frac{16.875}{4.5} = 3.75 \quad \text{or} \quad 3\theta^2 - 4\theta - 3.75 = 0$$

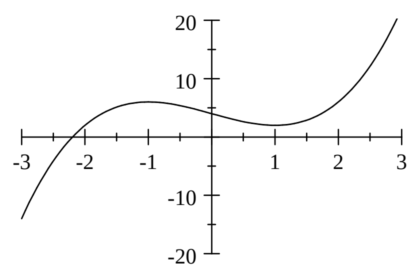
with solutions; $\theta_1 = \dfrac{4 + \sqrt{16 + 12 \times 3.75}}{6} = 1.96$ and $\theta_2 = \dfrac{4 - \sqrt{16 + 12 \times 3.75}}{6} = -0.64$ (to 2 decimal places). Both numbers are acceptable solutions.

Exercise 2

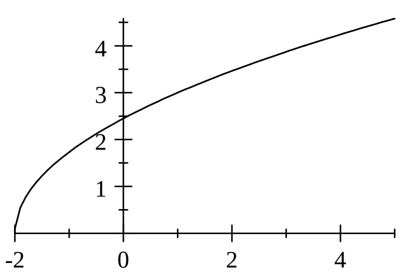
1. *The graphs of $f(x) = x^3$, $g(x) = x^3 - 3x + 4$, $h(x) = \sqrt{6 + 3x}$, and $u(x) = 4x + 1$ are given below.*



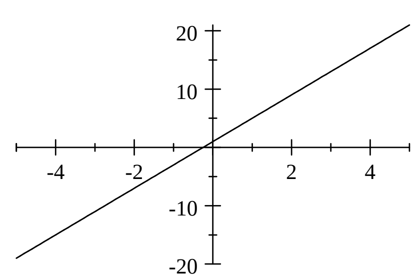
Graph of $f(x) = x^3$



Graph of $g(x) = x^3 - 3x + 4$



Graph of $h(x) = \sqrt{6 + 3x}$



Graph of $u(x) = 4x + 1$

(a) *Draw the cord joining the points* $(-3, f(-3))$ *and* $(0, f(0))$ *then find a number* θ *between* -3 *and* 0 *such that* $f(0) - f(-3) = 3f'(\theta)$.

(b) *Draw the cord joining the points* $(-2, g(-2))$ *and* $(2, g(2))$ *then find a number* θ *between* -2 *and* 2 *such that* $g(2) - g(-2) = 4g'(\theta)$.

(c) *Draw the cord joining the points* $(-2, h(-2))$ *and* $(3, h(3))$ *then find a number* θ *between* -2 *and* 3 *such that* $h(3) - h(-2) = 5h'(\theta)$

(d) *Draw the cord joining the points* $(1, u(1))$ *and* $(4, u(4))$ *then find a number* θ *between* 1 *and* 4 *such that* $u(4) - u(1) = 3u'(\theta)$

2. *Give an example of a function* f *with a smooth graph on an interval* $[a, b]$, *and with 3 or more numbers* θ *between* a *and* b *such that* $f(b) - f(a) = (b - a)f'(\theta)$.

3. *In this exercise, you have to show that* $\sqrt{1+x} \le 1 + \frac{1}{2}x$ *for all* $x \ge 0$. *To this end, consider the function* $g(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$, $x \ge 0$. *Show that* $g'(x) > 0$ *for all* $x > 0$. *Now take any* $x > 0$. *By the mean value theorem, there is a number* θ *between 0 and* x *such that* $g(x) - g(0) = xg'(\theta)$. *Explain why* $xg'(\theta) > 0$ *and deduce that* $g(x) - g(0) > 0$ *for all* $x > 0$. *Use this to show that* $\sqrt{1+x} \le 1 + \frac{1}{2}x$ *for all* $x \ge 0$.

Generalized Mean Value Theorem

Let f and g have derivatives on an interval $[a, b]$, and $a \le c < d \le b$. By the Mean Value Theorem, there are points θ and α in (c, d) such that

$$f(d) - f(c) = (d - c)\, f'(\theta) \quad \text{and} \quad g(d) - g(c) = (d - c)\, g'(\alpha)$$

There is no guarantee that θ is the same as α because f and g are different functions. Assume that $g'(\alpha) \ne 0$. Then we may divide to get

$$\frac{f(d) - f(c)}{g(d) - g(c)} = \frac{f'(\theta)}{g'(\alpha)} \tag{2}$$

The Generalized Mean Value Theorem asserts that if $g'(x) \ne 0$ for all x in (a, b) then, indeed, there is a **single point** β in (c, d) such that

$$\frac{f(d) - f(c)}{g(d) - g(c)} = \frac{f'(\beta)}{g'(\beta)} \tag{3}$$

Here is a very neat proof that uses the Mean Value Theorem.

Consider the function $h(x) = f(x) - \left(\frac{f(d) - f(c)}{g(d) - g(c)}\right) g(x)$. Direct computations, (do them), reveal that

$$h(c) = \frac{f(c)g(d) - f(d)g(c)}{g(d) - g(c)} = h(d)$$

Therefore, by the Mean Value Theorem, there is a point β in (c, d) such that

$$0 = h(d) - h(c) = (d - c)\, h'(\beta) \tag{4}$$

Substitute $h'(\beta) = f'(\beta) - \left(\frac{f(d) - f(c)}{g(d) - g(c)}\right) g'(\beta)$ into (4) then divide by $(d - c)$ to get

$$0 = f'(\beta) - \frac{f(d) - f(c)}{g(d) - g(c)} g'(\beta)$$

Since $g'(\beta) \ne 0$, (by hypothesis), we may divide by $g'(\beta)$ and re-arrange the resulting equation to get (3).

Exercise 3

1. *Let* $f(x) = x^2 + 4x$ *and* $g(x) = 2x^2 + x + 2$, $x \ge 0$. *Determine* $\frac{f(2)-f(0)}{g(2)-g(0)}$ *and* $\frac{f'(x)}{g'(x)}$ *then solve the equation* $\frac{f(2)-f(0)}{g(2)-g(0)} = \frac{f'(x)}{g'(x)}$ *for a number* β *in* $(0, 2)$ *such that* $\frac{f(2)-f(0)}{g(2)-g(0)} = \frac{f'(\beta)}{g'(\beta)}$.

2. *Let* $f(x) = x^2 + 4x$ *and* $g(x) = x^3 + x - 1$, *Find a number* β *in* $(-2, 1)$ *such that* $\frac{f(1)-f(-2)}{g(1)-g(-2)} = \frac{f'(\beta)}{g'(\beta)}$.

Some Applications of the Mean Value Theorem

We have already noted that if a function f is increasing on an interval $[a, b]$ then the tangents to its graph have positive slopes, therefore its derivatively $f'(x)$ cannot be negative. We can now verify the converse; that if a function g has a positive derivative on an interval $[a, b]$ then it must be increasing on $[a, b]$.

Claim 4 *If a function f has a positive derivative on an interval $[a, b]$, then it is increasing on the interval.*

To verify the claim, assume that $a \leq x < y \leq b$. By the mean value theorem, there is a number θ between x and y such that

$$f(y) - f(x) = (y - x)\, f'(\theta). \tag{5}$$

The right hand side of (5) is positive because $(y - x)$ and $f'(\theta)$ are both positive. Therefore $f(y)$ must be bigger than $f(x)$. Since x and y were arbitrary points in $[a, b]$, this proves that f is increasing on $[a, b]$.

Claim 5 *If a function f has a negative derivative on an interval $[a, b]$, then it is decreasing on the interval.*

To see this, assume that $a \leq x < y \leq b$. By the mean value theorem, there is a number θ between x and y such that

$$f(y) - f(x) = (y - x)\, f'(\theta).$$

This time $(y - x)\, f'(\theta)$ is negative, because $f'(\theta)$ is negative whereas $(y - x)$ is positive. Therefore $f(x)$ must be bigger than $f(y)$. Since x and y were arbitrary points in $[a, b]$, f is decreasing on $[a, b]$.

Claim 6 *If the derivative of a function f is zero on an interval $[a, b]$ then f is constant on $[a, b]$.*

For a proof, take any number x in $[a, b]$ which is bigger than a. By the mean value theorem, there is a number θ between x and a such that

$$f(x) - f(a) = (x - a)\, f'(\theta) = 0. \tag{6}$$

This implies that $f(x) = f(a)$. Since x was an arbitrary point in $[a, b]$, f has the constant value $f(a)$ on $[a, b]$.