Using Derivatives To Measure Rates of Change

A rate of change is associated with a variable $f(x)$ that changes by the same amount when the independent variable x increases by one unit. Here are two examples:

Example 1 *Consider the cost of renting a truck for one day when you are charged \$40.00 for the day plus 25 cents for every mile you drive. If you drive it for x miles then you are charged $f(x)$ dollars where*

$$f(x) = 40 + 0.25x$$

Clearly, $f(x)$ changes by 0.25 dollars whenever the mileage increases by one mile. In other words, when the independent variable changes from x to $x+1$ miles, $f(x)$ changes by 0.25 dollars, (from $40+0.25x$ to $40+0.25x+0.25$ dollars). In general, when it (the mileage) changes by h miles, then $f(x)$ changes by $0.25h$ dollars. The number 0.25 is called the rate of change of $f(x)$ with respect to x. It happens to be equal to $f'(x)$.

Example 2 *Say you leave home, drive to a highway and start cruising at a constant speed of $\frac{7}{6}$ miles per minute, (or 70 miles per hour). Suppose you start cruising when you are 3 miles from home. Then one minute later you will be $\frac{7}{6}+3$ miles from home, two minutes later you will be $\frac{14}{6}+3$ miles from home, x minutes later, you will be $f(x)$ miles from home where*

$$f(x) = \frac{7x}{6} + 3.$$

In this case, $f(x)$ changes by $\frac{7}{6}$ miles whenever x changes by one minute, therefore the rate of change of $f(x)$ with respect to x is $\frac{7}{6}$, (your cruising speed). This is also the derivative of $f(x)$ with respect to x.

In general, if a variable $f(x)$ changes by the same amount when the independent variable x increases by one unit, (from x to $x+1$) then the rate of change of $f(x)$ with respect to x is the difference between $f(x+1)$ and $f(x)$. This number also happens to be equal to $f'(x)$.

Instantaneous Rate of Change

If $f(x)$ does not change by the same amount whenever x increases by one unit then we cannot ask for a rate of change of $f(x)$ with respect to x. Instead, we ask for an **instantaneous rate of change**. The following example illustrates the idea.

Example 3 *Say you drop a stone from the top of a tall building and simultaneously start a stop watch. Let $f(x)$ be the distance, in feet, the stone has fallen, (from the point of release), when the stop watch reads x seconds. It can be shown, (e.g. experimentally), that before it hits the ground, $f(x)$ is given by*

$$f(x) = 16x^2$$

Because the term $16x^2$ is not linear, $f(x)$ does not change by the same amount whenever x changes by one unit. For example,

it changes by $f(1) - f(0) = 16$ feet when x changes from 0 to 1 seconds.
it changes by $f(2) - f(1) = 48$ feet when x changes from 1 to 2 seconds.
it changes by $f(3) - f(2) = 80$ feet when x changes from 2 to 3 seconds.

Therefore there is no number to pin down as the rate of change of $f(x)$ with respect to x. However, a rate of change at a specific time c is conceivable. Generalizing from Example 2 in which a rate of change of distance is speed, it should be the speed of the stone at time c. The technical term for it is ***the instantaneous rate of change of** $f(x)$ **with respect to*** *x at time c. To visualize it, imagine attaching a speedometer to the stone. Then the instantaneous rate of change of the distance, at time c, is the speedometer reading c seconds after the stone is released. We may calculate it as follows:*

Denote it by u and consider a time $c+h$ seconds, where h is a small number. Since the stone falls smoothly, its speed also changes smoothly. This implies that when h is close to 0, then its speed at time $c+h$ does not differ appreciably from u (its speed at time c). Therefore we can assume, with reasonable accuracy, that the speed is constant between the times c and $c+h$; and equal to u feet per second. (The smaller h is, the more reasonable the approximation.) This implies that the stone falls approximately uh feet between time c and time $c+h$ seconds. The actual distance is $f(c+h)-f(c)$, hence we must have $uh \simeq f(c+h)-f(c)$, or

$$u \simeq \frac{f(c+h)-f(c)}{h}$$

Note that this is true for any small number h, and that when h is close to 0, the quotient $\dfrac{f(c+h)-f(c)}{h}$ is close to $f'(c)$. Therefore we must have $u = \lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = f'(c) = 32c$.

In particular, its speed at time $x=3$ is 96 feet per second. However, because the speed is not constant, it is no longer true that when time changes by h seconds from 3 to $3+h$ seconds then the distance changes by $96h$ feet. The best we can now assert is that when h is close to 0 and time changes by h seconds from 3 to $3+h$ seconds, then the distance changes by ***approximately*** *$96h$ feet. This is because*

$$f(3+h)-f(3) \simeq f'(3)h,$$

when one appeals to differentials we encountered earlier. Alternatively, argue that the speed changes smoothly, therefore when h is very small, then the speed of the stone on the time interval $[3, 3+h]$ may be approximated by the speed at time 3, which is 96 feet per second. Therefore the distance travelled in these h seconds is approximately $96h$ feet.

In general, when time changes by a small amount h from x to $x+h$ seconds, then the distance changes by approximately $f'(x)h$ feet.

To generalize these observations, consider an arbitrary differentiable function $f(x)$. The **instantaneous rate of change of $f(x)$ with respect to x at a given point** c is defined to be $f'(c)$. It follows from the definition of $f'(c)$ that when h is close to 0 then

$$f(c+h)-f(c) \simeq f'(c)h.$$

In other words, when the independent variable changes by a small amount h, from c to $c+h$, the value of f changes by approximately $f'(c)h$.

Example 4 *The management of a grocery store estimate that when the price of a certain type of beef is set at x dollars per pound ($x>0$), the store sells $q(x)$ pounds of the beef daily, where*

$$q(x) = \frac{900}{x^2+9}$$

The derivative of $q(x)$ is $q'(x) = -1800x(x^2+9)^{-2}$. This is the instantaneous rate of change of $q(x)$ with respect to the price of the beef when the price is x dollars per pound. In particular, the instantaneous rate of change of $q(x)$ with respect to x when the price is 3 dollars, is

$$q'(3) = -\frac{50}{3}.$$

It follows that when the price changes by a small amount h from 3 to $3+h$ dollars per pound, the demand of the beef (i.e. the number of pounds sold per day), changes by approximately $-\frac{50}{3}h$ pounds. For example, if the price goes up by 45 cents from 3 dollars to 3.45, ($h=0.45$), dollars per pound, the demand changes by approximately

$$-\tfrac{50}{3} \times 0.45 = -7.5 \text{ pounds}$$

(i.e. it drops by about 7.5 pounds). If the price goes down by 36 cents from 3 dollars to 2.64 dollars per pound, ($h=-0.36$), the demand changes by approximately

$$\left(-\tfrac{50}{3}\right)(-0.36) = 6 \text{ pounds.}$$

In other words, the demand goes up by approximately 6 pounds.

Example 5 *A rock is blasted vertically up from the ground with an initial speed of 80 feet per second. As expected, it rises, and in the process, it slows down. When it reaches the highest point of its path, it stops momentarily then reverses direction and falls back to the ground. It can be shown that t seconds after being blasted up, its height $h(t)$, (in feet), above the ground is*

$$h(t) = 80t - 16t^2, \quad 0 \leq t \leq 5$$

Therefore its speed at time t is $h'(t) = 80 - 32t$. This enables us to deduce the following:

- *Its speed is zero when $h'(t) = 80 - 32t = 0$, therefore it reaches the highest point when $t = 2.5$.*
- *The maximum height it reaches must be $h(2.5) = 100$ feet, (corresponding to the time when its speed is zero).*
- *Since it takes 2.5 seconds to rise from the ground to the highest point, it must take 2.5 seconds to fall from the highest point to the ground. Therefore it spends 5 seconds in the air. Another way of obtaining this result is to use the formula $h(t) = 80t - 16t^2$ for the height of the rock above the ground. It is zero when $80t - 16t^2 = 0$. Solving gives $t = 0$ or $t = 5$. We know that when $t = 0$, it is just being blasted up. It follows that $t = 5$ corresponds to the instant it hits the ground on the way back, therefore it spends 5 seconds in the air.*

More Motion in a Straight Line

Consider an object that moves in a straight line. Suppose its distance, at time t, from some fixed point P is $s(t)$. The instantaneous rate of change of $s(t)$ at a specific time t is called the speed of the object. A common symbol for the speed at time t is $v(t)$. Therefore

$$s'(t) = v(t)$$

If the speed itself changes with time then the instantaneous rate of change of speed at time t is called the acceleration of the object. A common symbol for acceleration at time t is $a(t)$. Therefore

$$v'(t) = a(t).$$

For a specific example, consider a projectile that is projected vertically up with a speed of 40 feet per second, from a point P which is 68 feet above the ground. It is an established fact that if gravity is the only influence on its motion then its speed decreases by 32 feet per second in every passing second. (The number 32 is called the acceleration due to gravity.) If we use the above notations then $v'(t) = a(t) = -32$. It follows that

$$v(t) = -32t + c \tag{1}$$

where c is some constant to be determined from the given information. Indeed, since the speed of the projectile is 40 ft. per second at time $t = 0$, it follows that

$$40 = -32(0) + c$$

This implies that $c = 40$. Substituting this in (1) gives $v(t) = -32t + 40$. But $s'(t) = v(t) = -32t + 40$. It follows that

$$s(t) = -16t^2 + 40t + b$$

where b is another constant. To determine it, we use the fact that at time $t = 0$, the projectile was 68 ft. above ground. In other words, when $t = 0$, $s = 68$. Therefore

$$68 = -16\,(0)^2 + 40\,(0) + b \tag{2}$$

which gives $b = 68$. Substituting this in (2) gives $s(t) = -16t^2 + 40t + 68$. This is called the equation of motion for the projectile. The maximum height it reaches is obtained by determining the largest value of

$s(t)$. We do the standard thing: determine its critical point(s). Since $s'(t) = -32t + 40$, the critical point is the solution of the equation

$$-32t + 40 = 0,$$

which is $t = \frac{5}{4}$, and it is easily shown to be a point of relative maximum. Therefore the maximum height it reaches is $s(\frac{5}{4})$, which is equal to 93 feet.

If we want to know the time the projectile hits the ground, we solve the equation $s(t) = 0$ for positive time t. The positive solution is $t = 3.66$ seconds to 2 decimal places.

Exercise 6

1. *A rock thrown vertically up from the surface of the moon with a speed of 20 meters per second reaches a height of $h(t) = 20t - 2t^2$ meters in t seconds.*
   (a) *Find the rock's speed at time t.*
   (b) *How long does it take the rock to reach its highest point?*
   (c) *How high does the rock go?*
   (d) *How long does it take the rock to reach half its maximum height?*
   (e) *How many minutes elapse before it returns to the surface of the moon?*

2. *Channels in the cell membrane of a living cell offer resistance to the flow of sodium ions into the cell. The flow of sodium ions is measured as a current I in micro Amps. It is related to the membrane voltage v, measured in millivolts, by the formula*

$$I = 0.4\,(v + 40)$$

   (a) *Complete the table below*

Voltage (in millivolts).	-40	-20	0	10	15	20
Current (in micro Amps)						

   (b) *Draw a graph of I on a graph paper.*
   (c) *Use your graph to determine the derivative of I.*
   (d) *Determine the derivative of I.*
   (e) *What is the physical meaning of the derivative of I?*

3. *The number of gallons of water in a tank t minutes after the tank has started to drain is*

$$A(t) = 25\,(40 - t)^2\,. \quad 0 \le t \le 40$$

   (a) *How fast is the water running out at the end of 15 minutes?*
   (b) *At what time t is the water gushing out of the tank at the rate of 150 liters per minute?*

4. *The speed of blood, in centimeters per second, at a point x centimeters from the center of a given artery is given by the formula*

$$v = 1.28 - 20000x^2$$

   (a) *Complete the table below*

Distance from center of artery.	0.001	0.002	0.003	0.004	0.005	0.0053
Speed of blood						

(b) *Draw a graph showing how v is related to x.*

(c) *Interpret the derivative of v for $x = 0.001$ cm, 0.003 cm, and 0.007 cm.*

(d) *Compute the derivative of v when v is at its maximum. Explain your results.*

5. *An experimenter on top of a 120 feet building throws a stone vertically down. Assume that he releases it with a speed of 5 feet per second and that it moves under the influence of gravity. Let $s(t)$ be the distance of the stone from the ground, t seconds later. Show that $s(t) = -16t^2 - 5t + 120$, then calculate the time that elapses before the stone hits the ground.*

6. *An object moves along the x-axis in such a way that t seconds after starting to move, (assume that $t \geq 0$), it is a distance $x(t) = t^4 - 4t^3 + 4t^2$ feet from the origin. The object is moving forward (i.e. to the right) when its speed is positive, and it is moving backwards if the speed is negative. Give the time intervals during which it is moving, (i) forward, (ii) backwards. At what times does the object change direction?*

7. *The heart pumps blood through arteries, veins, and capillaries. Blood moves faster through narrow vessels and slower through wider vessels. The formula for the speed of blood flow is $S = \dfrac{Q}{A}$ where S is speed in centimeters per second, Q is flow rate in cubic centimeters per second and A is the cross sectional area of the vessel in square centimeters. For a blood flow rate $Q = 415$ cubic millimeters per second, use the given speed formula to find the blood speed for blood moving through an artery of cross section area a) $A = 1\ mm^2$, b) $A = 2\ mm^2$, c) $A = 3\ mm^2$, and d) $A = 4\ mm^2$ and make a table to show how speed and area are related.*

   (a) *Draw a speed and area relation on a graph paper.*

   (b) *Find and interpret the derivative of S with respect to A for $A = 1\ mm^2$, $A = 2\ mm^2$, $A = 3\ mm^2$, and $A = 4\ mm^2$.*

8. *Alveolus are lung air sacs through which gas exchanges between air and blood takes place. The liquid lining an alveolus creates a force called surface tension T. This force causes pressure on the gases in the alveolus. The formula for the pressure is $P = \dfrac{2T}{r}$ where P is the pressure, T is surface tension and r is the radius of the alveolus*

   (a) *For a surface tension of $T = 3$ units, construct a table for units of pressure caused for an alveolus radius of 1, 2, 3, and 4 then draw a pressure radius relation on a graph paper.*

   (b) *Determine and interpret the derivative of the pressure for a radius of 0.5 and a radius of 6.*

   (c) *What happens to the derivative of the pressure as the radius increases?*

9. *The trachea (wind pipe) is tube shaped with length L cm and radius r cm. The formula for its volume is*

$$V = \pi r^2 L$$

   (a) *Find the volume of a trachea with radius 1.25 cm and length 12 cm.*

   (b) *Sketch the graph of V as a function of positive r. Why do we consider only positive values of r?*

   (c) *Find and give a physical interpretation of the derivative of V with respect to r.*