Ordinary Annuities

Before defining an annuity, here is an example:

Example 1 *On January 1, Sarah learnt that she would be receiving $120 more per month, starting January 31, for the entire year. Instead of spending the extra money, she decided to deposit it, at the end of every month, into a savings account that pays an annual interest rate of 6.6%, compounded monthly. What will her investment be worth come December 31? (The answer is NOT* 12×120 *dollars because the deposits earn interest.)*

Solution:

We take one month at a time. The deposits earn compound interest at the rate of $\frac{6.6}{1200} = 0.0055$ *per month. The $120 she deposits into the account at the end of January will be in the account for 11 months, therefore come December 31 it will have grown to*

$$120\left(1+\frac{6.6}{1200}\right)^{11} = 120\,(1.0055)^{11} \quad \textit{dollars}$$

The $120 she deposits into the account at the end of February will be in the account for 10 months, therefore on December 31 it will have grown to

$$120\left(1+\frac{6.6}{1200}\right)^{10} = 120\,(1.0055)^{10} \quad \textit{dollars}$$

Continuing in this fashion, we calculate the amount each deposit will have grown to on December 31. We get the string of numbers

$$120\,(1.0055)^{11}\,,\ 120\,(1.0055)^{10}\,,\ 120\,(1.0055)^{9}\,,\ \ldots,\ 120\,(1.0055)\,,\ 120$$

Therefore, on December 31, the total amount of money in the account, (writing the terms in reverse order for convenience), will be

$$120 + 120\,(1.0055) + 120\,(1.0055)^2 + \cdots + 120\,(1.0055)^{10} + 120\,(1.0055)^{11}$$

Factoring out the 120 gives

$$120\left(1 + 1.0055 + 1.0055^2 + \cdots + 1.0055^{10} + 1.0055^{11}\right)$$

There is a quick way of determining the sum parentheses: use formula

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

with $x = 1.055$ *and* $n = 11$*. Actually, since* $x > 1$*, the two numbers* $1-x$ *and* $1-x^{n+1}$ *are negative. We might as well multiply the numerator and denominator of* $\frac{1-x^{n+1}}{1-x}$ *by* -1 *to get a quotient of two positive numbers. Therefore use the formula in the form*

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$$

to get

$$120\left(1 + 1.0055 + 1.0055^2 + \cdots + 1.0055^{10} + 1.0055^{11}\right) = \frac{120\left(1.0055^{12}-1\right)}{1.0055-1} = 1484.37 \ \textit{dollars.}$$

Sarah's twelve monthly payments in the above example are an annuity. In general, an annuity is a terminating stream of equal payments made at equal periods of time. If the payments are made at the end of each period and the interest on the payments is also calculated at the end of each period then the annuity is called an **ordinary annuity**. We plan to address only ordinary annuities in this section. The time between consecutive payments is called the **payment period**. The time from the beginning of the first payment period to the end of the last payment period is called the **term of the annuity**. The following is another example of an ordinary annuity:

Example 2 *A sports professional figured that he would have to retire after playing for 10 years. To prepare for the future, he decided to deposit \$85,000 at the end of each year, for 10 years, in an account that pays 7% interest compounded yearly. These payments form an annuity. The payment period is 1 year. The term of the annuity is 10 years. NOTE THE FOLLOWING:*

The first payment will earn interest for 9 years, because it is deposited at the end of the first payment period.

The second payment will earn interest for 7 years.

We hope the pattern is clear.

The last payment will not earn any interest because it is made right at the end of the term of the annuity.

AT THE END OF THE TERM:

1. *The the first payment will have grown to* $85,000\,(1.07)^9$ *dollars.*
2. *The the second payment will have grown to* $85,000\,(1.07)^8$ *dollars.*
3. *The the third payment will have grown to* $85,000\,(1.07)^7$ *dollars.*
4. *The the fourth payment will have grown to* $85,000\,(1.07)^6$ *dollars.*
5. *The the fifth payment will have grown to* $85,000\,(1.07)^5$ *dollars.*
6. *The the sixth payment will have grown to* $85,000\,(1.07)^4$ *dollars.*
7. *The the seventh payment will have grown to* $85,000\,(1.07)^3$ *dollars.*
8. *The the eighth payment will have grown to* $85,000\,(1.07)^2$ *dollars.*
9. *The the ninth payment will have grown to* $85,000\,(1.07)$ *dollars.*
10. *The last payment earns no interest therefore it will be* $85,000$ *dollars*

The total amount in the bank after the 8 payments will be

$$85,000\left(1 + 1.07 + 1.07^2 + 1.07^3 + \cdots + 1.07^9\right)$$

Just as we did above, we use the formula $1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ *with* $x = 1.07$ *and* $n = 9$ *to add the ten numbers and the result is*

$$85,000\left(1 + 1.07 + 1.07^2 + 1.07^3 + \cdots + 1.07^9\right) = 85,000\left(\frac{1.07^{10} - 1}{1.07 - 1}\right) = 1174398.08$$

One more example:

Example 3 *A student borrowed money to pay his tuition. He borrowed* \$8000 *on May 1, 2010, another* \$8000 *on May 1, 2011, another* \$8000 *on May 1, 2012, and another* \$8000 *on May 1, 2013. How much will he be owing on May 1, 2014, just before graduating if the interest rate is 4.8% per year compounded annually?*

The four payments of \$8000 *each to the student form an annuity.*

- *The* \$8000 *he borrowed on May 1, 2010 will have accumulated to* $8000\left(1+\frac{4.8}{100}\right)^4 = 8000\,(1.048)^4$ *dollars on May 1, 2014.*
- *The* \$8000 *he borrowed on May 1, 2011 will have accumulated to* $8000\left(1+\frac{4.8}{100}\right)^3 = 8000\,(1.048)^3$ *dollars on May 1, 2014.*
- *The* \$8000 *he borrowed on May 1, 2012 will have accumulated to* $8000\left(1+\frac{4.8}{100}\right)^2 = 8000\,(1.048)^2$ *dollars on May 1, 2014.*
- *The* \$8000 *he borrowed on May 1, 2013 will have accumulated to* $8000\left(1+\frac{4.8}{100}\right) = 8000\,(1.048)$ *dollars on May 1, 2014.*

Therefore, on May 1, 2014, the amount of money he will be owing, (to the nearest cent), is

$$\begin{aligned}8000\left(1.048^4+1.048^3+1.048^2+1.048\right) &= 8000\,(1.048)\left(1.048^3+1.048^2+1.048+1\right)\\ &= 8000\,(1.048)\left(\frac{1.048^4-1}{1.048-1}\right) = 36028.79 \textit{ dollars}\end{aligned}$$

It is NOT $4\times 8000 = 32000$ *dollars because he is charged interest.*

A New Term: We found that the 12 deposits Sarah made had a total value of 1966.27 dollars on December 31. This figure is called the **future value** of the annuity. In the example of the professional sportsman, the ten deposits had a total value of 1174398.08 dollars. This is the **future value of the annuity**. In general, the total value of an annuity, (i.e. the total value of a set of payments including the interest earned), at a specified time in the future, (e.g. at the end of the term of the annuity), is called the future value of the annuity.

Future Value of an Ordinary Annuity

Consider an **ordinary annuity** consisting of n payments of P dollars. There are n payment periods and one payment is made at the end of each period. Let the interest rate per period be j. To get the future value of the annuity at the end of term, we simply evaluate the accumulated value of every payment then add up.

The **first** payment will earn interest for a total on $(n-1)$ periods. Therefore at the end of term of the annuity, it will have grown to $P\,(1+j)^{n-1}$ dollars.

The **second** payment will earn interest for a total on $(n-2)$ periods. Therefore at the end of term of the annuity, it will have grown to $P\,(1+j)^{n-2}$ dollars.

Now the pattern should be clear. The last but one payment, (i.e. the $(n-1)$-th payment), will earn interest for 1 period, therefore it will grow up to $P\,(1+j)$ by the end of the term. Of course the n-th payment earns no interest therefore the total value of the n payments is

$$P\,(1+j)^{n-1} + P\,(1+j)^{n-2} + \cdots + P\,(1+j) + P$$

When we write it in reverse order and use the formula $1+x+x^2+\cdots+x^n = \dfrac{x^{n+1}-1}{x-1}$, we conclude that the future value of the annuity is

$$\begin{aligned}FV \text{ of annuity } &= P + P(1+j) + \cdots + P\,(1+j)^{n-2} + P\,(1+j)^{n-1}\\ &= P\left[1 + (1+j) + \cdots + (1+j)^{n-2} + (1+j)^{n-1}\right]\\ &= P\left[\frac{(1+j)^n-1}{(1+j)-1}\right] = P\left[\frac{(1+j)^n-1}{j}\right]\end{aligned}$$

Conclusion: The future value of an annuity consisting of n payments of P dollars each, made at the end of each one of n equal periods is

$$FV \text{ of annuity } = P\left[\frac{(1+j)^n - 1}{j}\right]$$

where j is the interest rate per period.

Example 4 *In early January, Tina got a baby boy and, starting from January 31 of the same month, she deposited \$95.00 into a savings fund that pays interest at the rate of 5.4% per year, compounded monthly. She plans to hand over the account to the son at the end of the month in which he turns 21 years. How much money will be in the account when it is turned over to the young man?*

The first step is to determine the interest rate per month. Since the rate is 5.4% per year, it translates into a monthly rate j equal to $(5.4\%) \div 12$, which simplifies to $j = 0.0045$ per month.

Next, we need to know the total number of deposits she will make over the 21 year period. They are $n = 21 \times 12 = 252$ in total.

Since she deposits $P = 95$ dollars per period, the future value of the 252 payments is

$$FV = 95\left[\frac{(1+0.0045)^n - 1}{0.0045}\right] = 44336.87 \textit{ dollars}$$

Note that if Tina had simply locked up the monthly deposit in a safe in her house, they would amount to just

$$95 \times 252 = 23940 \textit{ dollars}$$

This illustrates the power of compounding.

Exercise 5

1. *Calculate the future value of the given ordinary annuity:*
   (a) *Payments of \$30000 made at the end of each year starting December 31, 2000 for 15 years. The annual interest rate is 4.2%.*
   (b) *Payments of \$757 each made at the end of each month starting October 31, 2010 for 8 years. The annual interest rate is 8.4% compounded monthly.*
   (c) *Payments of \$1200 each made at the end of every quarter of a year, from March 31, 2000 to December 31, 2014. The annual interest rate is 3.6% compounded quarterly.*
2. *What monthly payments into an account that earns annual interest of 6.3%, compounded monthly, will amount to at least \$15580 in 8 years? The payments are to be made at the end of each month.*
3. *At what monthly interest rate will 36 monthly deposits of \$200 each, (deposited at month-ends), accumulate to \$8500?*
4. *On January 31, 2002, Jane began depositing \$65.00 every end of month into a savings account that pays an annual interest rate of 3.3%, compounded monthly. How much money will be in the account on December 31, of 2014?*
5. *Jonathan used to spend \$108 per month on cigarettes. He quit smoking and decided to deposit \$108 at the end of every month into a savings account that pays interest at the rate of 3% per year, compounded monthly. Calculate the amount of money that will be in the account 25 years later.*
6. *Say you can afford to part with \$2400 every year. There is a savings account that pays interest at the rare rate of 12% per year. It is December 31, 2012.*
   (a) *Say you decide to deposit \$2400 into the account at the end of every year starting December 31, 2013. How much money will be in the account on December 31, 2015?*

(b) *Now assume that instead of making \$2400 deposits every end of year, you decide to make \$200 deposits every month-end starting December 31 of 2012, (NOT December 31, 2013). (Of course you will end up depositing \$2400 into the account every year.) How much money will be in the account on December 31, 2015, assuming that interest is compounded monthly?*

(c) *Instead of making \$2400 payments every end of year, you decide to make* $\$\frac{2400}{365}$ *deposits every end of day starting December 31 of 2012. (Again, you will end up depositing \$2400 into the account every year.) How much money will be in the account on December 31, 2015, assuming that interest is compounded daily?*

(d) *In general, suppose you divide a year into* n *equal intervals and you decide to deposit* $\frac{2400}{n}$ *dollars at the end of every one of these* n *intervals starting with the period that begins on December 31, 2012. (Of course, you will end up depositing \$2400 into the account every year.) Assuming that interest is compounded at the end of each interval, show that on December 31, 2015, the account will contain* $20000\left[\left(1+\frac{0.12}{n}\right)^{3n}-1\right]$ *dollars.*