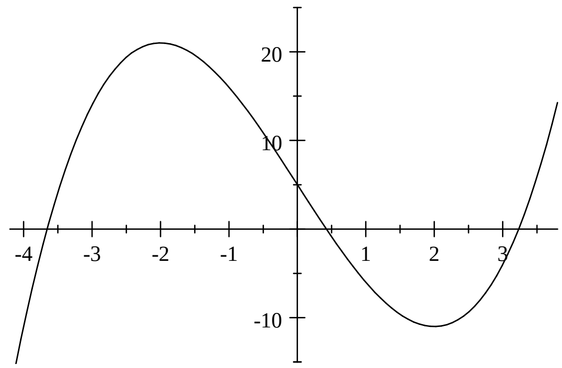
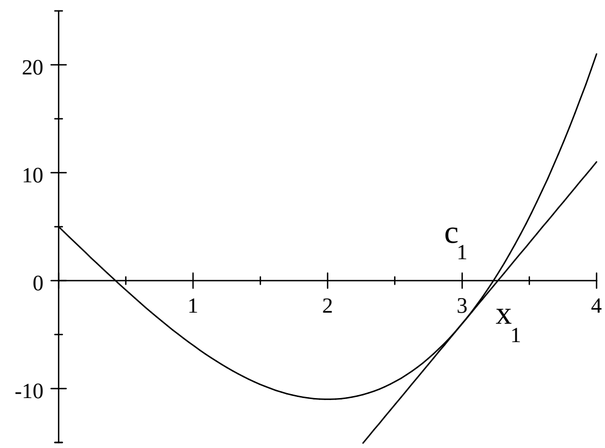
Newton's Method for Calculating Approximate Roots

The following example introduces the essential ideas of Newton's method.

Example 1 *Consider the function* $f(x) = x^3 - 12x + 5$. *Its graph is given below.*



The numbers c such that $f(c) = 0$ *are called the solutions, or roots, of the equation* $x^3 - 12x + 5 = 0$. *They are the x-intercepts of the graph of f. To the nearest whole number, they are* $c_1 \simeq 3$, $c_2 \simeq 0$ *and* $c_3 \simeq -4$. *Newton's method is a procedure for calculating a better approximate solution, given an approximate solution. For example, the procedure enables us to calculate a better approximate solution of* $x^3 - 12x + 5 = 0$, *given the approximate solution* $c_1 = 3$. *The procedure is pretty straight-forward. We draw the tangent to the graph of f at* $(3, f(3)) = (3, -4)$ *then determine its x-intercept, which we may denote by* x_1. *The graph below of a magnified section of the curve and its tangent at* $(3, -4)$, *shows clearly that* x_1 *is a better approximate solution than 3.*



x_1 *is nearer the solution than 3*

In practice we do not draw the tangent. We simply determine its equation then calculate its x-intercept. Since $f'(x) = 3x^2 - 12$, *its slope is* $f'(3) = 3(3)^2 - 12 = 15$, *therefore its equation is given by*

$$y + 4 = 15\,(x - 3) \quad \textit{or} \quad y = 15x - 49$$

Its x-intercept may be obtained by solving the equation

$$15x - 49 = 0$$

The result is $x_1 = \frac{49}{15}$. *Since* $f(\frac{49}{15}) = 0.66$, *(to 2 decimal places), which is closer to zero than* $f(3) = -4$, $x_1 = \frac{49}{15}$ *is definitely a better approximate solution of the given equation than* 3.

To generalize, suppose f is a given function and x_0 is an approximate solution of the equation $f(x) = 0$. (We are assuming that x_0 has been determined by some means, e.g. from a sketch of the graph of f.) Consider the tangent to the graph of f at $(x_0, f(x_0))$. Its slope is $f'(x_0)$, therefore its equation is

$$y = f(x_0) + (x - x_0)\, f'(x_0).$$

Denote its x-intercept by x_1. Then under suitable conditions, x_1 is a better approximate solution of the equation $f(x) = 0$ than x_0. You may determine x_1 by solving the equation

$$f(x_0) + (x - x_0)\, f'(x_0) = 0$$

Remove parentheses and rearrange to get $xf'(x_0) = x_0 f'(x_0) - f(x_0)$. Now divide by $f'(x_0)$, and you should get

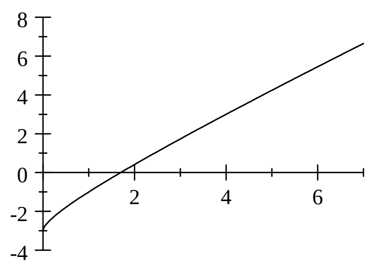
$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We have therefore shown that:

- *If x_0 is an approximate solution of the equation $f(x) = 0$, then $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is generally a better approximate solution of the equation than x_0. This procedure for calculating better approximate solutions is called Newton's method.*

Exercise 2

1. *The graph of $f(x) = \sqrt{x} + x - 3$ below shows that the equation $\sqrt{x} + x - 3 = 0$ has a solution close to 2. Use $x_0 = 2$ as an approximate solution*



to calculate a better approximate solution. You can actually solve this equation using the quadratic formula. Solve it and compare the result to the approximate solution.

2. *Use $x_0 = -4$ as an approximate solution to the equation $x^3 - 12x + 5 = 0$ of Example 1 to calculate a better approximate solution of the equation.*

3. *Let $f(x) = x^2 - 15$. We may view the square root of 15 as a solution to the equation $f(x) = 0$. Take $x_0 = 4$ as an approximate value of $\sqrt{15}$ and use Newton's method to calculate a better approximate value of $\sqrt{15}$.*

4. *Use a suitable function and follow the steps in question 3 above to determine an approximate value of $\sqrt[3]{29.2}$ with the help of Newton's method.*

5. *Show that if x_0 is a root of the equation $f(x) = 0$, then applying Newton's method to x_0 does not provide anything new.*

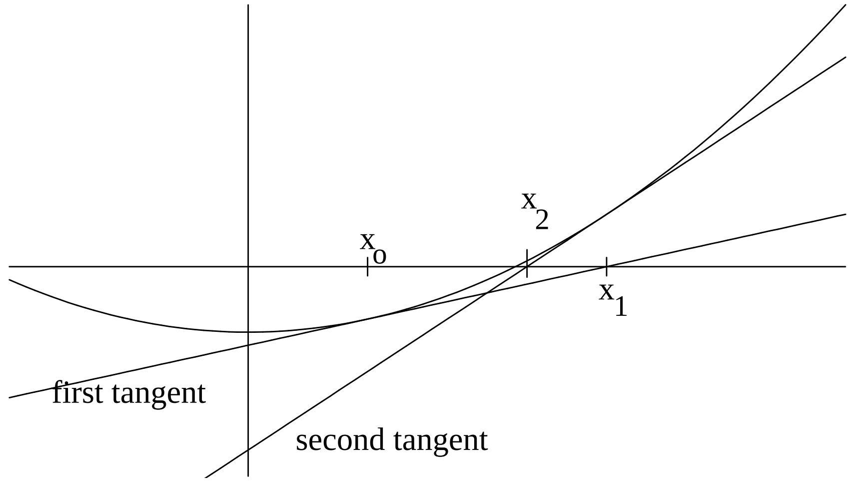
Repeated application of Newton's method

Consider a function f and the equation $f(x) = 0$. By Newton's method, if x_0 is an approximate solution of $f(x) = 0$ then a better approximate solution is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This is called the first iterate, or the first approximation given by Newton's method. It may be used to get an even better approximate solution x_2, called the second iterate, or the second approximation. As you would expect, it is the x-intercept of the tangent at $(x_1, f(x_1))$ and it is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$



This may be used to get a third approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

In general, if you have an nth approximation x_n, you may calculate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

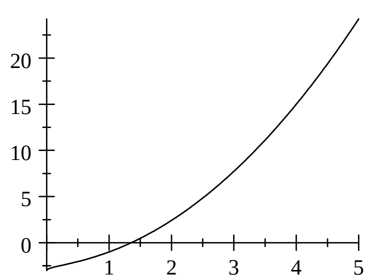
Example 3 *Consider the approximate solution $x_1 = \frac{49}{15}$ for the equation $x^3 - 12x + 5 = 0$ in Example 1. We may use it to get an even better approximate solution x_2 given by*

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{49}{15} - \frac{f(\frac{49}{15})}{f'(\frac{49}{15})} = \frac{49}{15} - \frac{(\frac{49}{15})^3 - 12(\frac{49}{15}) + 5}{3(\frac{49}{15})^2 - 12} = 3.2337 \text{ to 4 dec. pl.}$$

Since $f(3.2337) = (3.2337)^3 - 12(3.2337) + 5 = 0.009804$ and $f(x_1) = f(\frac{49}{15}) = 0.66$, x_2 is definitely a better approximate solution than x_1. (Use x_2 to get x_3.)

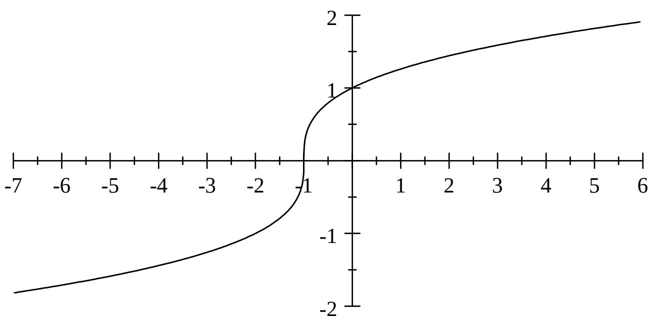
Exercise 4

1. *The graph of $f(x) = \sqrt{x} + x^2 - 3$ is given below. It shows that $x_0 = 1$ is an approximate solution of the equation $f(x) = 0$. Use it to determine the first and second approximations x_1, and x_2 respectively, given by Newton's method. Round off x_2 to 3 dec. pl.*



2. *Sketch the graph of* $f(x) = x^3 + 3x^2 + 2$ *and use it to verify that the equation* $x^3 + 3x^2 + 2 = 0$ *has one real root. Use the sketch to estimate the root then use Newton's method to determine two better approximations.*

3. *There is no guarantee that Newton's method will always work. For example, consider the equation* $(x+1)^{1/3} = 0$*. Of course we know its solution; it is* $x = -1$*. Pretend that you do not know it. The graph of* $f(x) = (1+x)^{1/3}$ *is given below.*



Take $x_0 = 0$ *as an approximate solution and determine* x_1*. Next, use* x_1 *to determine* x_2*. Also, draw the tangents at* $(x_0, f(x_0))$ *and* $(x_1, f(x_1))$*. Can you see why it fails in this case?*

4. *Say you want to calculate the root of* $x^3 - 12x + 5 = 0$ *in Example 1, which is close to 3, accurately to 5 decimal places. You compute* x_1*,* x_2*,* x_3*, ... and stop when there is no change in the first five decimal places of your iterate. Recall that we found that* $x_1 = \frac{49}{15} = 3.26666667$*, and* $x_2 = 3.23374046$*. We calculate* x_3 *using the equation*

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.233193843.$$

There is a change in the fourth decimal place so we compute x_4*. It turns out to be*

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3.233193694.$$

Since there is no change in the first 5 decimal places, we stop. We give the root as 3.23319 *correct to 5 decimal places.*

Use a similar procedure to determine the root of $\sqrt{x} + x^2 - 3 = 0$ *that is between 1 and 2, correct to 5 decimal places.*

5. *Consider the function* $f(x) = x^3 + x - 3$*. Since* $f(1) = -1$ *and* $f(2) = 7$*, the graph of* f *crosses the* x*-axis between* $x = 1$ *and* $x = 2$*. It follows that the equation* $x^3 + x - 3 = 0$ *has a root between* 1 *and* 2*. We may take* $x = 1$ *or* $x = 2$ *as an approximate root of this equation. But of the two,* $x = 1$ *is probably a better approximation because* $f(1)$ *is closer to 0 than* $f(2)$*. Use it to calculate the root correct to 2 decimal places.*

6. *Show that the equation* $x^4 + x^3 + 4x - 1 = 0$ *has a root between* 0 *and* 1 *then use Newton's method to calculate it accurately to 2 decimal places.*