More General Compounding

There are cases in which interest is calculated more frequently than yearly as in the above examples. For example, credit card companies do not wait a whole year to charge interest on credit card debts. They generally charge it monthly. Some institutions charge it semiannually, (i.e. every 6 months), or quarterly, (i.e. every 3 months). But in all these cases, the idea is the same; (i) each year is divided into a specified number of periods, (ii) interest is calculated at the end of every period, and it is added to what is owed at the beginning of the period, to earn interest during the subsequent periods. In most of the cases we are going to encounter, the interest rate will be quoted as an **annual interest rate**, also called **Annual Percentage Rate**, which is abbreviated to **APR** and is denoted by i.

Example 1 *Suppose the $3500 inherited by Devon in the exercise above was deposited into a savings account that pays interest at the rate of 3% per year, calculated monthly, (technically "compounded monthly"). This means that at the end of every month, interest is calculated and added to the account. The quoted 3% interest rate is for a whole year. To get the rate per month, we divide the quoted rate by 12. Therefore the interest per month is* $(3 \div 12)\,\%$ *which is equal to* $\frac{1}{4}$ *% or* 0.0025 *per month. It follows that at the end of the first month, interest equal to* 3500×0.0025 *dollars is added to the account to bring the new amount, in dollars, to*

$$3500 + 3500 \times 0.0025 = 3500\,(1 + 0.0025) = 3500\,(1.0025)$$

At the end of the second month, interest amounting to $3500\,(1.0025) \times 0.0025$ *is added to the account to bring the new amount, in dollars, to*

$$3500\,(1.0025) + 3500\,(1.0025) \times 0.0025 = 3500\,(1.0025)\,(1 + 0.0025) = 3500\,(1.0025)^2 .$$

The value of the investment increases by a factor of 1.0025 *every month. Therefore one year later, (which is 12 months later), it will amount to* $3500\,(1.0025)^{12} = 3606.45$ *dollars to the nearest cent. Two years later, (i.e. 24 months later), it will have grown to* $3500\,(1.0025)^{24}$

$$3500\,(1.0025)^{24}$$

Three years later, its value will be

$$3500\,(1.0025)^{36}$$

In general, n years, the value will be

$$3500\,(1.0025)^{12n}$$

The money grows by a factor of $(1.0025)^{12}$ *every year, instead of a factor* 1.03 *if interest were compounded yearly. To 4 decimal places,* $(1.0025)^{12} = 1.0304$ *which is a little higher than the* 1.03 *growth factor when interest is compounded yearly. The table below shows the amounts, to the nearest cent, in the account at different times under the two ways of compounding the interest*

End of year	*Amount in account when interest is compounded yearly*	*Amount in account when interest is compounded monthly*
1	*3605.00*	*3606.46*
2	*3713.15*	*3716.15*
3	*3824.54*	*3829.18*
5	*4057.46*	*4065.66*
10	*4703.71*	*4722.74*

To generalize, take the case of $\$P$ deposited into a savings account that pays interest at an annual rate i. Suppose the interest is compounded k times a year. This means that:

- (i) a year is divided into k equal periods,
- (ii) interest is calculated at the end of each period and is added to what was in the account at the beginning of the period, to earn interest in subsequent periods.

Since i is the quoted interest rate per year, the rate for each of the k periods into which the year is divided should be i/k. Therefore, at the end of the first period, the P dollars will earn interest equal to $P \times \frac{i}{k}$ dollars. This will be added to the P dollars, therefore the account will contain

$$P + P \times \frac{i}{k} = P\left(1 + \frac{i}{k}\right) \text{ dollars.}$$

At the end of the second period, it will contain

$$P\left(1 + \frac{i}{k}\right) + P\left(1 + \frac{i}{k}\right) \times \frac{i}{k} = P\left(1 + \frac{i}{k}\right)^2 \text{ dollars.}$$

At the end of each period, the value increases by a factor $\left(1 + \frac{i}{k}\right)$, therefore at the end of the nth period, the dollar amount in the account will be

$$P\left(1 + \frac{i}{k}\right)^n \tag{1}$$

Example 2 *A developer borrowed \$80000 to buy land. She is charged interest at an annual rate of 6%, compounded quarterly. She plans to pay it back, with the interest, in one lump sum. How much, to the nearest dollar, will she owe two years later?*

> ***Solution:*** *Using formula (1), we have $P = 80000$, $i = 0.06$ and $k = 4$. Since there are 8 quarters in two years, $n = 8$, therefore she owes*
>
> $$80000\left(1 + \frac{0.06}{4}\right)^8 = 90119 \textit{ dollars.}$$

Example 3 *Angela would like to have \$21,000 available five years from now for a down payment on a house. She would like to know how much money, to the nearest dollar, she should invest now in a savings account paying an annual interest rate of 8%, compounded monthly, to accumulate the \$21,000 in 5 years time. Let the amount be P dollars.*

> *(a) How much will her investment be worth at the end of: (i) the first year, (ii) the second year, (iii) the fifth year?*
>
> *(b) Use the expression you get in (iii) and the fact that she is aiming for \$21,000 to determine P.*
>
> ***Solution:***
>
> *(a) In this problem, $i = 0.08$ and $k = 12$. (i) Interest is compounded 12 times in one year, therefore at the end of the first year her investment will be worth*
>
> $$P\left(1 + \frac{0.08}{12}\right)^{12} \textit{ dollars.}$$
>
> *(ii) Interest is compounded 24 times in two years, therefore at the end of the second year, the account will be worth*
>
> $$P\left(1 + \frac{0.08}{12}\right)^{24} \textit{ dollars.}$$
>
> *(iii) Interest is compounded 60 times in five years, therefore at the end of the fifth year, the account will be worth*
>
> $$P\left(1 + \frac{0.08}{12}\right)^{60} \textit{ dollars.}$$

(b) We must find P such that $P\left(1+\frac{8}{1200}\right)^{60} = 21,000$. *A calculator gives* $\left(1+\frac{0.08}{12}\right)^{60} =$ 1.489846. *Therefore we have to find P such that*

$$1.489846P = 21000 \tag{2}$$

Solving (2) gives $P = \frac{21000}{1.489846} = 14,095$ *dollars to the nearest dollar.*

Exercise 4

1. *You deposit $1000 into a savings account that earns interest at an annual rate of 5.1%, compounded monthly. Therefore the interest rate per month is* $\frac{5.1}{1200} = 0.004\,25$. *Complete the following table*

# of months since making deposit	0	1	2	3	4	x
Value of investment	1000	1000 (1.004 25)				

How much money will be in the account 10 years later?

2. *Vonneh secured a $20,000 loan for tuition and is not required to start paying it back till after she has graduated, (4 years later). She is charged interest at the rate of 6% per year, compounded monthly.*

   (a) *What is the interest rate per month?*

   (b) *Complete the table below that shows how the amount of money she owes changes as the months go by.*

# of months later	*0*	*1*	*2*	*3*	x
Amount she owes	*20000*				

   (c) *What will she be owing when she graduates 4 years after securing the loan? Round off your answer to the nearest dollar.*

3. *You deposit $400 into a savings account with an annual interest rate of* 3.6% *compounded monthly.*

   (a) *What is the interest rate per month?*

   (b) *Complete the following table that shows how the money in the account increases as the months go by.*

# of months since making deposit	0	1	2	3	4	x
Value of investment	400					

   (c) *How much money is in the account 4 years later?*

4. *You have $2000 to put away in a savings account for 2 years. One account pays an annual interest rate of 16% compounded every 6 months. Another one pays an annual rate of 15.6% compounded monthly. Which of the two will yield a higher return at the end of 2 years and by how much will it be higher?*

5. *You need $4000 four years from now. How much money do you have to put into a savings account that pays annual interest at the rate of 4.5% compounded daily, to have the $4000 available at the time?*

6. *What APR will grow $3000 into $4000 in seven years if the interest is compounded monthly?*

7. *Sam has just filed his tax return and is expecting a refund cheque of $550.00 in two months time. The tax service firm that did his taxes makes him the following offer: It would give him a cheque of $480.00 to save him the two months of waiting for his refund cheque. We may look at this offer as a loan of $480.00 to Sam which he must pay back in two months with his $550.00 cheque. Assume that he accepts the offer and that interest is compounded monthly.*

   (a) *What is the implied interest rate per month? Round off your answer to 3 decimal places.*

   (b) *What is the implied interest rate per year? Round off your answer to 2 decimal places.*