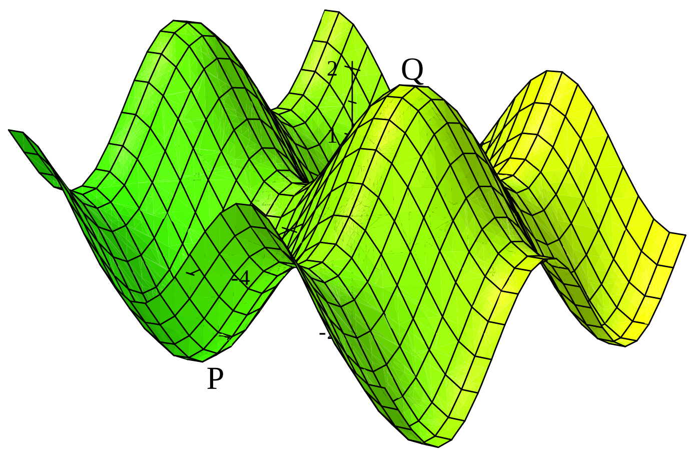
# Maxima/Minima

**Example 1** *The figure below shows a graph of $f(x, y) = \sin x + \cos y$.*




*Consider the point $(-\frac{1}{2}\pi, \pi)$ in its domain and the point $P$ on its graph with coordinates*

$$(-\tfrac{1}{2}\pi, \pi, f(-\tfrac{1}{2}\pi, \pi)) = (-\tfrac{1}{2}\pi, \pi, -2)$$

*In the vicinity of $P$, the graph resembles a right-face-up bowel. This is because $f\ (-\frac{1}{2}\pi, \pi)$ is the smallest value of $f(x, y)$ in the vicinity of $(-\frac{1}{2}\pi, \pi)$. A more precise way of saying this is that we can find a disc $D$ centred at $(-\frac{1}{2}\pi, \pi)$ such that $f(-\frac{1}{2}\pi, \pi) \leq f(x, y)$ for all $(x, y)$ in $D$. For this reason, $(-\frac{1}{2}\pi, \pi)$ is called a point of relative minimum for $f$.*

*The graph has the opposite shape near the point $Q$ with coordinates $(\frac{1}{2}\pi, 0, f(\frac{1}{2}\pi, 0)) = (\frac{1}{2}\pi, 0, 2)$. It is shaped like a face-down bowel because $f(\frac{1}{2}\pi, 0)$ is the largest value of $(x, y)$ in the vicinity of $(\frac{1}{2}\pi, 0)$. More precisely, we can find a disc $D$ centred at $(\frac{1}{2}\pi, 0)$ such that $f(x, y) \leq f(\frac{1}{2}\pi, 0)$ for all $(x, y)$ in $D$. For this reason, $(\frac{1}{2}\pi, 0)$ is called a point of relative maximum for $f$. These examples suggest the following definition:*

**Definition 2** *Let $f(x, y)$ be a function of two variables and $(c, d)$ be a point in its domain.*

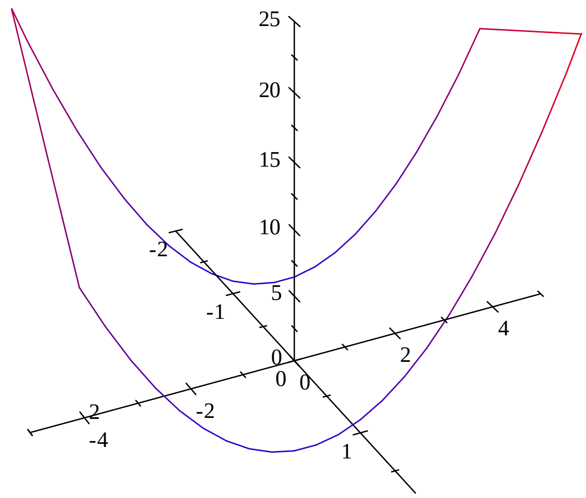
*We say that $(c, d)$ is a point of relative minimum for $f$ if there is a disc $D$ centred at $(c, d)$ such that $f(c, d) \leq f(x, y)$ for all $(x, y)$ in $D$.*

*We say that $(c, d)$ is a point of relative maximum for $f$ if there is a disc $D$ centred at $(c, d)$ such that $f(x, y) \leq f(c, d)$ for all $(x, y)$ in $D$.*

*An extreme point for $f$ is any point of relative maximum or any point of relative minimum for $f$.*

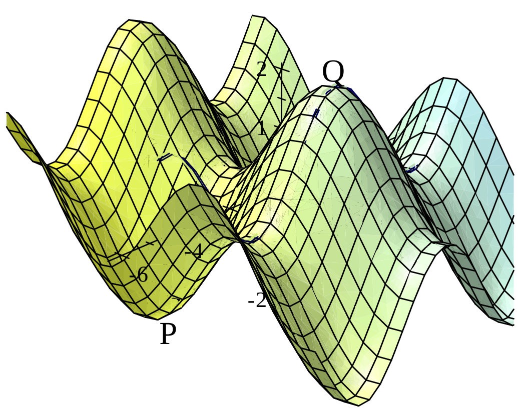
**Example 3** *Let $f(x, y) = (x + y)^2$. Its graph resembles a piece of paper that is smoothly folded into half.*

*Every point on the line $y = -x$, $z = 0$ is a point of relative minimum for $f$.*

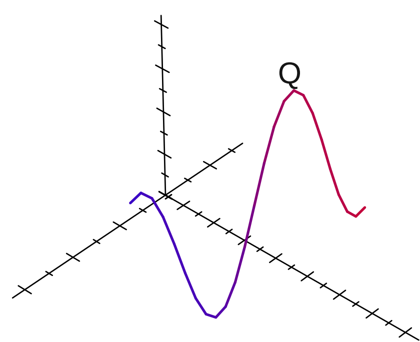



We locate extreme points of a function of two variables the same way we locate them for a function of one variable. Recall that at an extreme point $c$ of a function of one variable, the tangent to the graph of $f$ is horizontal, hence $f'(c) = 0$. Therefore we look for its extreme points among the numbers $x$ such that $f'(x) = 0$. We called them, (i.e. the numbers $x$ such that $f'(x) = 0$), the critical points of $f$. It may be necessary to apply tests to each critical point in order to establish its nature, (i.e. to establish whether it is a point of relative maximum/minimum or neither).

Before handling a general function of two variables, we examine the extreme points of $f(x, y) = \sin x + \cos y$. We pointed out, in Example 1 above that $(\frac{1}{2}\pi, 0)$ is a point of relative maximum for $f$. Its graph is shown below with the $x = \frac{1}{2}\pi$ section highlighted. The section is also drawn separately.




Graph of $f$ with $x = \frac{1}{2}\pi$ section highlighted




Graph of the $x = \frac{1}{2}\pi$ section

Clearly, the curve has a point of relative maximum at $y = 0$. This implies that the tangent to the curve at 0 is horizontal, therefore $f_y(\frac{1}{2}\pi, 0) = 0$. A similar argument yields $f_x(\frac{1}{2}\pi, 0) = 0$.

Turning to an arbitrary function $f(x, y)$ of two variables, we note that if $(c, d)$ is a point of relative maximum (minimum) for $f$ then the number $d$ must be a point of relative maximum (minimum) for the $x = c$ section of $f$. In other words, $d$ is a point of relative maximum (minimum) for $v(y) = f(c, y)$, therefore $v'(d) = 0$. Since $v'(d) = f_y(c, d)$, it follows that $f_y(c, d) = 0$. A similar argument applies to the $y = d$ section of $f$. The number $c$ is a point of relative maximum (minimum) for the function $u(x) = f(x, d)$, therefore $u'(c) = 0$. Since $u'(c) = f_x(c, d)$, it follows that $f_x(c, d) = 0$.

**Conclusion 4** *Let $f(x, y)$ be a function of two variables. If $(c, d)$ is a point of relative maximum or a point of relative minimum for $f$ then $f_x(c, d) = 0$ and $f_y(c, d) = 0$. Therefore, if you are looking for the extreme points of $f$, look among the points $(c, d)$ such that $f_x(c, d) = 0$ and $f_y(c, d) = 0$.*

**Definition 5** *Let $f(x, y)$ be a function of two variables. A point $(c, d)$ in its domain such that $f_x(c, d) = 0 = f_y(c, d)$ is called a critical point of $f$.*

**Example 6** *Let $f(x, y) = \sin x + \cos y$. Then $f_x(x, y) = \cos x$ and $f_y(x, y) = -\sin y$. The critical points of $f$ are the points $(c, d)$ such that $f_x(c, d) = 0$ and $f_y(c, d) = 0$. Since $f_x(x, y) = 0$ when $x = \frac{1}{2}\pi$, $\frac{1}{2}\pi \pm \pi$, $\frac{1}{2}\pi \pm 2\pi$, ... and $f_y(x, y) = 0$ when $y = 0$, $\pm\pi$, $\pm 2\pi$, ... its critical points are $(\frac{1}{2}\pi, 0)$, $(\frac{1}{2}\pi, \pi)$, ...., $(\frac{1}{2}\pi + n\pi, m\pi)$, ... where $m$ and $n$ are integers.*

**Example 7** *Let $f(x, y) = x^2 + y^2 + xy + 3x$. Then $f_x(x, y) = 2x + y + 3$ and $f_y(x, y) = 2y + x$. The critical points of $f$ are the pairs$(x, y)$ such that*

$$2x + y + 3 = 0$$

$$2y + x = 0$$

*The first equation implies that $y = -3 - 2x$. Substituting in the second equation gives*

$$-6 - 4x + x = 0$$

*Solving for $x$ gives $x = -2$. Substituting in $y = -3 - 2x$ gives $y = 1$. Therefore $f$ has one critical point, namely $(-2, 1)$.*

## A Test for the Nature of a Critical Point

Let $(c, d)$ be a critical point for a function $f(x, y)$ which has continuous partial derivatives. To simplify notation, denote $f_{xx}(c, d)$ by $A$, $f_{xy}(c, d)$ by $B$ and $f_{yy}(c, d)$ by $C$. We must demand that at least one of these three numbers is non-zero, (else the test does not apply). Form the number

$$H = AC - B^2$$

1. If $H$ and $A$ are both positive then $(c, d)$ is a point of relative minimum. (Actually, if $H > 0$ then $A$ and $C$ must have the same sign. Therefore $(c, d)$ is a point of relative minimum if $H$ and $A$ or $C$ are positive.)

2. If $H$ is positive but $A$ (or $C$) is negative then $(c, d)$ is a point of relative maximum.

3. If $H$ is negative then $(c, d)$ is neither a point of relative maximum nor a point of relative minimum. Thus every disc centred at $(c, d)$ contains pairs $(x, y)$ such that $f(x, y) < f(c, d)$ and pairs $(r, s)$ such that $f(r, s) > f(c, d)$.

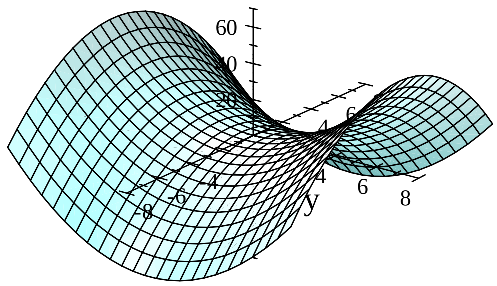
4. If $H = 0$ then the test is inconclusive. In other words, $(c, d)$ could be a point of relative maximum or a point of relative minimum or neither.

This is proved ahead, (see page **??**) after the introduction of Taylor's theorem for a function of two variables.

**Example 8** *Consider the critical point $(\frac{1}{2}\pi, 0)$ of the function $f(x, y) = \sin x + \cos y$ in Example 1. Since $f_{xx}(x, y) = -\sin x$, $f_{xy}(x, y) = 0$ and $f_{yy}(x, y) = -\cos y$, it follows that $f_{xx}(\frac{1}{2}\pi, 0) = -1$, $f_{xy}(\frac{1}{2}\pi, 0) = 0$ and $f_{yy}(\frac{1}{2}\pi, 0) = -1$. Therefore $H = (-1)(-1) - 0 = 1$ which is positive. Since $f_{xx}(\frac{1}{2}\pi, 0)$ is negative, $(\frac{1}{2}\pi, 0)$ is a point of relative maximum for $f$.*

**Example 9** *Let $f(x, y) = x^2 + y^2 + xy + 3x$ be the function in Example 7. We found that $(-2, 1)$ is a critical point for $f$. It turns out that $f_{xx}(x, y) = 2$, $f_{xy}(x, y) = 1$ and $f_{yy}(x, y) = 2$. Therefore $A = f_{xx}(-2, 1) = 2$, $B = f_{xy}(-2, 1) = 1$ and $C = f_{yy}(-2, 1) = 2$, hence $H = 4 - 1 = 3$ which is positive. Since $f_{xx}(-2, 1)$ is positive, $(-2, 1)$ is a point of relative minimum for $f$.*

**Example 10** *Let $f(x,y) = x^2 - y^2$. Its critical points are the pairs $(x,y)$ such that $f_x(x,y) = 0$ and $f_y(x,y) = 0$. Since $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$, the function has one critical point, namely $(0,0)$. It turns out that $f_{xx}(x,y) = 2$, $f_{xy}(x,y) = 0$ and $f_{yy}(x,y) = -2$. Therefore $A = f_{xx}(0,0) = 2$, $B = f_{xy}(0,0) = 0$ and $C = f_{yy}(0,0) = -2$, hence $H = -4$, which is negative. It follows that $(0,0)$ is neither a point of relative maximum nor a point of relative minimum. Its graph below supports this conclusion.*

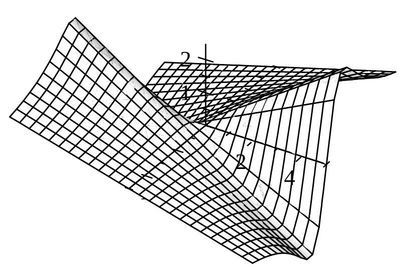



**Example 11** *Let $f(x,y) = x^4 + y^6$, $g(x,y) = -x^4 - y^4$ and $h(x,y) = x^4 - y^4$. You can easily show that $(0,0)$ is a critical point, and that $H = 0$ for each of the three functions. Since $x^4 + y^6 \geq 0$ for all $(x,y)$ and $f(0,0) = 0$, $(0,0)$ is a point of relative minimum for $f$. One shows in a similar way that $(0,0)$ is a point of relative maximum for $g$. Finally, every disc centred at $(0,0)$ contains a following point of the form $(x,0)$ where $h$ is positive and a point of the form $(0,y)$ where $h$ is negative, therefore $(0,0)$ is neither a point of relative maximum nor a point of relative minimum for $h$. This demonstrates that when $H = 0$, the test is inconclusive.*

**Exercise 12**

1. *Show that $f(x,y) = x^2 - xy + \frac{1}{3}y^3$ has four critical points and establish the nature of each one.*
2. *Determine the critical points, if any, of each given function and establish the nature of each one.*

| | | |
|---|---|---|
| *a)* $f(x,y) = 2x^2 + 2xy + y^2 - 4x$ | *b)* $f(x,y) = \frac{1}{3}x^3 + 4xy - 2y^2$ | *c)* $f(x,y) = x^3 - 3xy + \frac{1}{8}y^3$ |
| *d)* $f(x,y) = x^3 - 3x^2y - 3x^2 - 3y^2$ | *e)* $f(x,y) = x^4 - 2x^2y^2 + 2y^3$ | *f)* $f(x,y) = xy - y^3 - x^3$ |
| *g)* $f(x,y) = x^2 + y^3 - 2xy - y + 1$ | *h)* $f(x,y) = y^2 - x^2y + x^2 - y$ | *i)* $f(x,y) = \dfrac{xy}{1+x^2}$. |




*Graph of* $f = \dfrac{xy}{1+x^2}$.