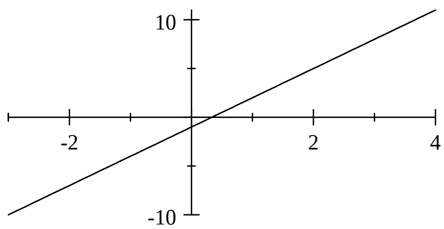
Information Provided by Derivatives

We start with a number of "new" terms. We assume that all the functions in this section are differentiable.

- *We say that a function f is* **increasing** *on an interval $[a, b]$ if its values get bigger as the independent variable x increases. Thus f is increasing if $a \le x_1 < x_2 \le b$ implies that $f(x_1) \le f(x_2)$.*

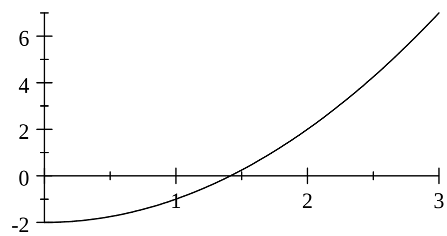
Example 1 *Let $f(x) = 3x - 1$. It graph is given below. It is increasing on any interval $[a, b]$. For if $x_1 < x_2$ then $3x_1 < 3x_2$, therefore $3x_1 - 1 < 3x_2 - 1$. In other words, if $x_1 < x_2$ then $f(x_1) \le f(x_2)$.*



Example 2 *The function $f(x) = x^2 - 2$, $x > 0$ is increasing on any interval $[a, b]$ with $a \ge 0$. To see this, take any x_1 and x_2 with $0 < x_1 < x_2$. Then*

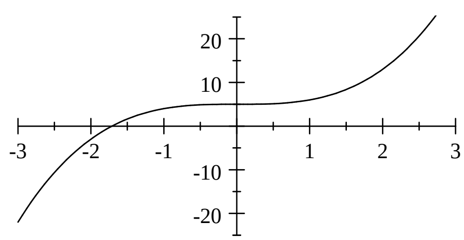
$$f(x_2) - f(x_1) = x_2^2 - x_1^1 = (x_2 - x_1)(x_2 + x_1)$$

Since $(x_2 + x_1) > 0$, because it is a sum of two positive numbers, and $(x_2 - x_1) > 0$ because $x_1 < x_2$, their product $(x_2 - x_1)(x_2 + x_1)$ must be positive. This proves that $f(x_1) \le f(x_2)$.



Example 3 *The function $f(x) = x^3 + 5$ is increasing on any interval $[a, b]$. To see this take any two numbers x_1 and x_2 with $x_1 < x_2$. Then*

$$f(x_2) - f(x_1) = x_2^3 - x_1^3 = (x_2 - x_1)\left(x_2^2 + x_1 x_2 + x_1^2\right)$$



Note that $x_2^2 + x_1 x_2 + x_1^2 = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, therefore it cannot be negative. It follows that $(x_2 - x_1)\left(x_2^2 + x_1 x_2 + x_1^2\right)$ is non-negative which proves that $f(x_1) \le f(x_2)$.

The graphs in examples 1, 2 and 3 are all "forward-leaning", so they have tangents with nonnegative slopes. This is another way of saying that they have non-negative derivatives. In general, if a function f is increasing on an interval then the quotient

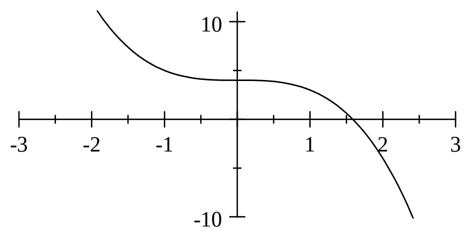
$$\frac{f(x+h) - f(x)}{h}$$

cannot be negative. This implies that $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, which is $f'(x)$, cannot be negative. The converse is also true; that if the derivative of a function is positive on an interval then the function is increasing on the interval. This will be proved using the Mean Value Theorem. Therefore, to show that a given function f is increasing on an interval $[a, b]$, simply **Show that $f'(x)$ is never negative on $[a, b]$.**

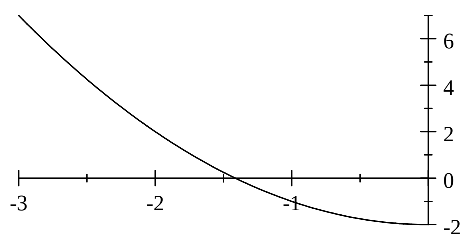
A decreasing function has the opposite properties; its values decrease as the independent variable increases. More precisely:

- *A function $g(x)$ is* **decreasing** *on an interval $[a, b]$ if $g(x_2) \leq g(x_1)$ whenever $a \leq x_1 < x_2 \leq b$.*

Example 4 *Let $g(x) = 4 - x^3$. Then $g(x_2) - g(x_1) = -(x_2 - x_1)\left(x_2^2 + x_1 x_2 + x_1^2\right)$ which is negative when $x_1 < x_2$. This proves that g is decreasing on any interval $[a, b]$.*



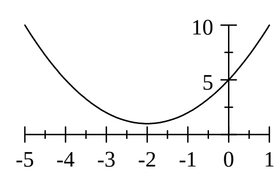
Example 5 *$f(x) = x^2 - 2$ is decreasing on any interval $[a, b]$ with $b \leq 0$. For let $x_1 < x_2 < 0$. Then $f(x_2) - f(x_1) = (x_2 - x_1)(x_2 + x_1)$ is negative because it is a product of a positive number $x_2 - x_1$ and a negative number $x_1 + x_2$. It follows that $f(x_2) \leq f(x_1)$.*



Examples 4 and 5 show that if a function is decreasing on a given interval then its graph is "back-leaning", suggesting that its derivative cannot be positive. It will also be proved that if the derivative of a given function is negative on an interval then the function is decreasing on the interval. Therefore, to show that a function f is decreasing on an interval $[a, b]$, simply **Show that $f'(x)$ is never positive on $[a, b]$.**

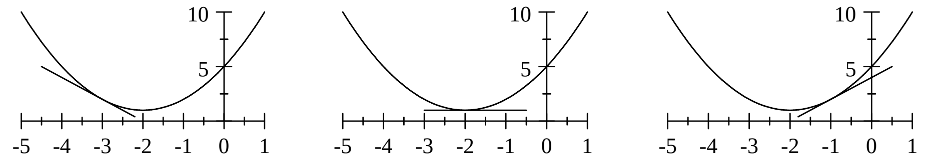
- We say that a number c is a **point of relative minimum** for a given function f if the graph of f near $(c, f(c))$ resembles a right-side up bowel.

Example 6 *Let $f(x) = x^2 + 4x + 5$. Its graph near $c = -2$ resembles a right-side up bowel, therefore $c = -2$ is a point of relative minimum for f.*



Remark 7 *The term "relative minimum for f" is used to point out that the values of f at all near-by points x are bigger than its value at c. As the graph in Example 6 shows, $f(x)$ is decreasing to the immediate left*

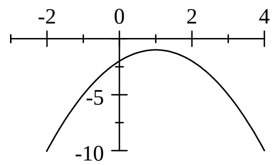
of c, and it is increasing to the immediate right of c. Therefore $f'(x)$ is negative at all near-by points x to the left of c and it is positive at the near-by points x to the right of c. At the point c itself, $f'(c) = 0$.



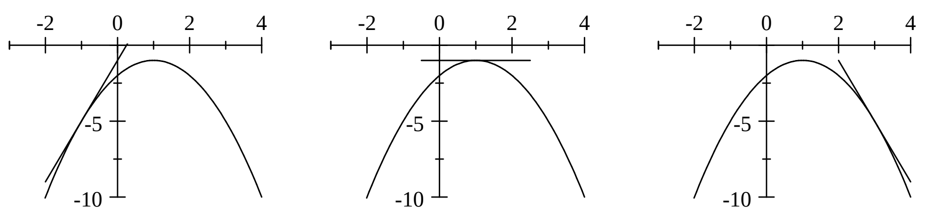
Negative slope to the left *Zero slope at the point* *Positive slope to the right*

- We say that a number c is a **point of relative maximum** for a given function f if the graph of f near $(c, f(c))$ resembles an upside-down bowel.

Example 8 *Let $f(x) = -2 + 2x - x^2$. Its graph near $c = 1$ resembles an upside-down bowel, therefore $c = 1$ is a point of relative maximum for f.*



Remark 9 *The term "relative maximum" is used to point out that the values of f at all near-by points x are smaller than its value at c. As the graph in Example 8 shows, $f(x)$ is increasing to the immediate left of c, and it is decreasing to the immediate right of c. Therefore $f'(x)$ is positive at all near-by points x to the left of c and it is negative at the near-by points x to the right of c. At the point c itself, $f'(c) = 0$.*



Positive slope to the left *Zero slope at the point* *Negative slope to the right*

We define points of relative maximum/minimum without reference to graphs as follows:

Definition 10 *Let f be a given function and c be a point in its domain.*

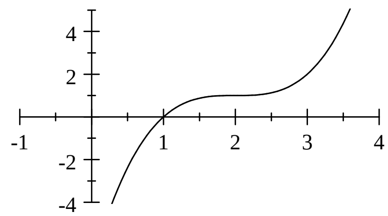
1. *We say that c is a point of relative maximum for f if there is an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).*
2. *We say that c is a point of relative minimum for f if there is an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).*

Every maximization (minimization) problem we have solved so far came down to determining a point of relative maximum (minimum) for some function f. We did this by determining the *numbers c such that* $f'(c) = 0$. The technical term for such a number is a *critical point of* f. More precisely;

- A **critical point** for a given function f is a number c such that $f'(c) = 0$.

In particular, a point of relative maximum for a differentiable function f is a critical point of f, and so is any point of relative minimum. But, as the examples below show, a function can have a critical point that is neither a point of relative minimum nor a point of relative maximum.

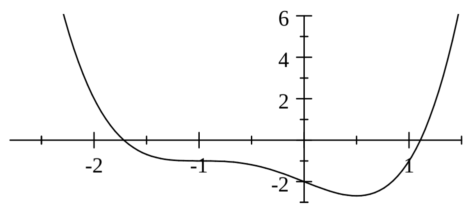
Example 11 *Let* $f(x) = (x-2)^3$. *Then* $f'(x) = 3(x-2)^2$ *which is zero when* $x = 2$. *As its graph below shows,* $c = 2$ *is a critical point of* f *that is neither a point of relative maximum nor a point of relative minimum.*



Example 12 *Let* $h(x) = (x+1)^3(x-1) - 1 = x^4 + 2x^3 - 2x - 2$. *Then*

$$h'(x) = 4x^3 + 6x^2 - 2 = 2(x+1)^2(2x-1).$$

Therefore h *has critical points* $c_1 = -1$ *and* $c_2 = \frac{1}{2}$. *It should be clear from the graph below that* c_1 *is neither a point of relative maximum nor a point of relative minimum for* h.



The following are steps you may follow to locate the critical points of a given function f and establish their nature without first drawing its graph:

- Step 1: *Solve the equation* $f'(x) = 0$. *The solutions (if any), are the critical points* c *of* f.
- Step 2: *For each critical point* c, *do the following: (i) Pick a point* x_1 *to the left of* c. *If there are other critical points of* f *to the left of* c *then* x_1 *must be a point between* c *and the next critical point to the left of* c. *(ii) Pick a point* x_2 *to the right of* c. *If there are other critical points of* f *to the right of* c *then* x_2 *must be a point between* c *and the next critical point to the right of* c. *Now evaluate* $f'(x_1)$ *and* $f'(x_2)$.

  If $f'(x_1) > 0$ and $f'(x_2) < 0$ *then* c *is a point of relative maximum.*

  If $f'(x_1) < 0$ and $f'(x_2) > 0$ *then* c *is a point of relative minimum.*

  If $f'(x_1)$ *and* $f'(x_2)$ *have the same sign then* c *is neither a point of relative maximum nor a point of relative minimum for* f.

Example 13 *Consider* $h(x) = x^4 + 2x^3 - 2x - 2$ *in Example 12. We found that* $h'(x) = 4x^3 + 6x^2 - 2 = 2(x+1)^2(2x-1)$. *Its critical points are* $c_1 = -1$ *and* $c_2 = \frac{1}{2}$. *To establish the nature of* c_1 *we evaluate* h' *at a convenient point* x_1 *to left of* -1 *and a convenient point* x_2 *between* -1 *and* $\frac{1}{2}$. *We may take* $x_1 = -2$ *and* $x_2 = 0$. *Since* $h'(-2) = -4$ *and* $h'(0) = -2$ *which are both negative,* c_1 *is neither a point of relative maximum nor a point of relative minimum. In the case of* $c_2 = \frac{1}{2}$, *we may take* $x_3 = 1$ *to the right of* $\frac{1}{2}$. *We already have* $x_2 = 0$ *to the left* $\frac{1}{2}$. *Since* $h'(1) = 4$ *which is positive, and* $h'(0)$ *is negative,* c_2 *must be a point of relative minimum.*

A visual method which is particularly useful when one is dealing with a function with many critical points, is to draw a table that shows how the sign of the derivative changes on the different intervals determined by the critical points of the given function. In the case of $h(x) = x^4 + 4x^3 - 2x - 2$ of Example 12, the intervals are $(-\infty, -1)$, $(-1, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$. They are listed in the first row of the table below. The second and third rows show the signs of the factors $2(x+1)^2$ and $(2x-2)$ of $h'(x)$ on these intervals. The fourth row shows the sign of $h'(x) = 2(x+1)^2(2x-1)$, and the last row shows the way the graph leans on each interval.

For example, on the interval $x < -1$, the derivative of h is negative, therefore its graph leans backwards as indicated by the left-leaning line segment $\backslash$.

	$x < -1$	$x = -1$	$-1 < x < \frac{1}{2}$	$x = \frac{1}{2}$	$\frac{1}{2} < x$
$2(x+1)^2$	$++++$	0	$+++++$	$+$	$++++$
$(2x-1)$	$----$	$-$	$-----$	0	$++++$
$2(x+1)^2(2x-1)$	$----$	0	$-----$	0	$++++$
Shape of graph	$\backslash$	—	$\backslash$	_	/

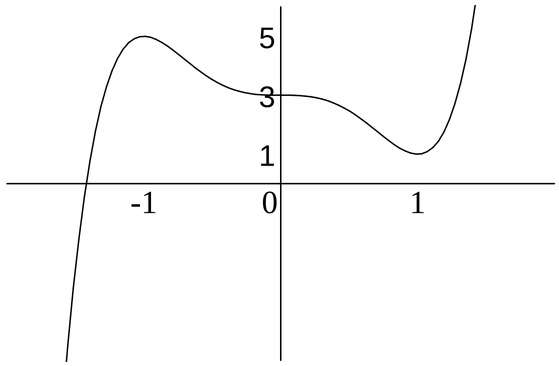
It is now clear that $c_1 = -1$ is neither a point of relative maximum nor a point of relative minimum, and $c_2 = \frac{1}{2}$ is a point of relative minimum.

Example 14 *Consider $f(x) = x^2 - 5x + 1$. Its derivative is $f'(x) = 2x - 5 = 0$, which is zero when $x = \frac{5}{2}$. This is its only critical point. Since $f'(x)$ is negative when $x < \frac{5}{2}$ and is positive when $x > \frac{5}{2}$, it follows that $c = \frac{5}{2}$ is a point of relative minimum for f.*

Example 15 *Let $f(x) = 3x^5 - 5x^3 + 3$. Its derivative is $f'(x) = 15x^4 - 15x^2 = 15x^2\,(x-1)\,(x+1)$. Its critical points are $c_1 = -1$, $c_2 = 0$ and $c_3 = 1$. The table below shows how the sign of $f'(x)$ changes on the different intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ determined by the critical points.*

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x$
$15x^2$	$+++$	$+$	$++++$	0	$++++$	$+$	$+++$
$(x-1)$	$---$	$-$	$----$	$-$	$----$	0	$+++$
$(x+1)$	$---$	0	$++++$	$+$	$++++$	$+$	$+++$
$h'(x)$	$+++$	0	$----$	0	$----$	0	$+++$
Shape of graph	/	—	$\backslash$	—	$\backslash$	_	/

It follows from the last row that $c_1 = -1$ is a point of relative maximum, $c_2 = 0$ is neither a point of relative maximum nor a point of relative minimum, and $c_3 = 1$ is a point of relative minimum. The values of f at its critical points are $f(-1) = 5$, $f(0) = 3$ and $f(1) = 1$. This information, plus the last row of the above *table suggest that the graph of f has the shape shown below. It is called a sketch of the graph of f.*

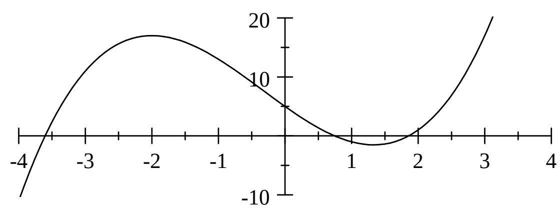


A sketch of the graph of f

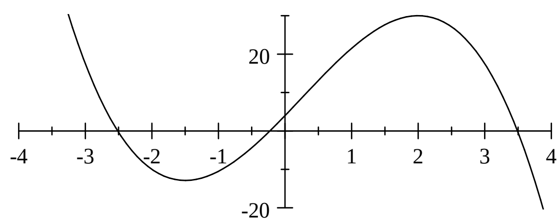
Exercise 16

1. *In (a) to (c), the formula of a function f and its graph are given. Use the graph to estimate its critical points then determine their exact values by solving an appropriate equation. Also state the nature of each critical point.*

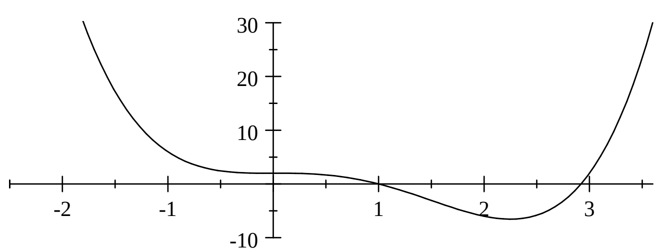
   (a) $f(x) = x^3 + x^2 - 8x + 5$



   (b) $f(x) = 4 + 18x + \frac{3}{2}x^2 - 2x^3$



   (c) $f(x) = 2 + x^4 - 3x^3$



2. *In Example 15, we determined the critical points of the function* $f(x) = 3x^5 - 5x^3 + 3$*, established their nature, then gave a sketch of its graph. Do the same for each of the following functions:*

a. $f(x) = x^3 - 27x + 1$

b. $g(x) = x^3 - x^2 - 8x + 9$

c. $h(x) = x^3 - 3x^2 + 3x + 5$

d. $v(x) = \left(x^3 - 9x\right)^{5/3}$, *(5 critical points).*

e. $f(x) = \sqrt{x^3 - 12x + 25}$

f. $u(x) = x^2 - \frac{3}{2}x^{4/3} + 2$, *(3 critical points).*

g. $w(x) = \sqrt{8 + 2x - x^2}$,

h. $f(x) = x^4 - 10x^2$

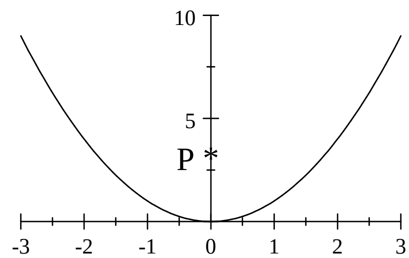
i. $g(x) = e^x - x$

j. $h(x) = x + \cos x, \quad 0 \le x \le 2\pi$

k. $u(x) = \frac{1}{2}x - \sin x, \quad -\pi \le x \le 2\pi$

l. $w(x) = xe^x$

3. *In this exercise you have to determine the shortest distance from the point* $P\,(0,3)$ *to the parabola* $y = x^2$.

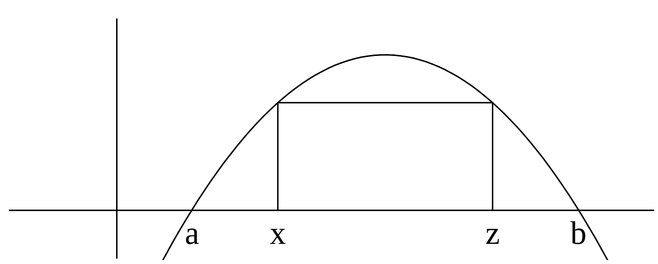


A point on the parabola has the form (x, x^2). *Denote its distance from* $(0,3)$ *by* $D(x)$. *Then*

$$D(x) = \sqrt{x^2 + (x^2 - 3)^2} = \sqrt{x^4 - 5x^2 + 9}$$

Therefore you have to find the smallest possible value of $\sqrt{x^4 - 5x^2 + 9}$ *as* x *varies among the real numbers. To avoid the square root, note that it is sufficient to determine the smallest value of* $(D(x))^2 = x^4 - 5x^2 + 9$. *Denote it by* s. *The required number is* $\sqrt{s}$. *Determine the three critical points of* $f(x) = x^4 - 5x^2 + 9$*, establish their nature then describe how* $D(x)$ *changes with* x*, and find the required distance.*

4. *Let* $a < b$ *and* R *be the region enclosed by the graph of* $f(x) = -(x-a)(x-b)$*, (a parabola), and the* x*-axis. Let* $a < x < \frac{a+b}{2}$. *A rectangle is drawn inside* R *with one vertical side through* $(x, 0)$ *and the other vertical side through* $(z, 0)$ *as shown below.*



(a) *Show that* $z = a + b - x$ *and that the area* $A(x)$ *of the rectangle is*

$$A(x) = (x-a)(x-b)(2x-a-b)$$

(b) *Take* $a = 0$ *and* $b = 4$ *and determine the critical points of the resulting function. Also establish the nature of each critical point then describe how the area changes as* x *changes from 0 to 2. What is its maximum value?*

5. *Consider the sum of a non-zero number and its reciprocal. Of course the sum depends on the number you choose. For example, the sum of* 5 *and its reciprocal* $\frac{1}{5}$ *is* 5.2*, whereas the sum of 10 and its reciprocal is* 10.1*. Find a positive number* x *such that the sum of* x *and its reciprocal is as small as possible.*