Exponents in Some Financial Transactions

Exponents play a big role in financial transactions involving **compound interest**. If you are not familiar with interest, simply view it as the price you pay for the privilege of using someone else's money. You will run into it when you borrow money to buy a big ticket item like a car or a house. Considering that the lender has no access to the money you have borrowed, (until you have paid it back), it is no surprise that you pay some price for using it. What you borrow is generally called the **principal**. The interest you are charged depends on the size of the principal and on the time it takes you to pay it back. The bigger the principal or the longer you take to pay it back, the more interest you pay. In many cases the interest is charged at regular time intervals, e.g. monthly, semi-annually, yearly, etc. Say it is charged yearly. If you divide the interest charged at the end of the first year by the amount you borrowed, the result is called the **annual interest rate.** For example, if you borrow $400 and you are charged $56 interest one year later then the annual interest rate is

$$\frac{56}{400} = 0.14$$

This may be given as a percentage, which is 14%.

In practice the annual interest rate is specified by the lender and we will generally call it the **quoted rate**. Then to calculate the interest you will be charged in a year, simply multiply the rate by the amount you owed at the beginning of the year.

Compound interest is interest which is "**compounded**". In other words, the interest is calculated at the agreed time intervals *and it is added to what you already owe.* From the moment it is added, it too earns interest. Here is an example:

Example 1 *A student borrows $20,000 for tuition and she is charged interest at the rate of 6% per year,* ***compounded yearly****. This means that at the end of every year, interest is calculated and it is added to what she owed at the beginning of the year. The interest is 6% of what she owed at the beginning of the year. Therefore:*

At the end of year 1, the interest which is 6% *of 20000, is added to 20000. Now she owes*

$$20000 + 200000.06 = 20000\,(1 + 0.06) = 20000\,(1.06)$$

To get a useful pattern, keep this as $20000\,(1.06)$ *dollars. If you multiply to get 21200 dollars you will miss a useful formula.*

At the end of year 2, the interest, which is 6% of $20000\,(1.06)$*, is added to* $20000\,(1.06)$*. Then she will be owing*

$$20000\,(1.06) + 20000\,(1.06) \times 0.06 = 20000\,(1.06)\,(1 + 0.06) = 20000\,(1.06)\,(1.06) = 20000\,(1.06)^2$$

At the end of year 3, the interest, which is 6% of $20000\,(1.06)^2$*, is added to* $20000\,(1.06)^2$*. She will be owing*

$$20000\,(1.06) + 20000\,(1.06)^2 \times 0.06 = 20000\,(1.06)^2\,(1 + 0.06) = 20000\,(1.06)^2\,(1.06) = 20000\,(1.06)^3$$

The pattern should be clear.

# of years later	*0*	*1*	*2*	*3*	*4*	x
Amount she owes	*20000*	$20000\,(1.06)$	$20000\,(1.06)^2$	$20000\,(1.06)^3$	$20000\,(1.06)^4$	$20000\,(1.06)^x$

Exercise 2

1. *Devon inherited \$3500 and the money was deposited into a savings account that pays interest at the rate of 3% per year, compounded yearly. Complete the following table that shows how the amount of money in the account changed as the years passed by:*

# of years later	0	1	2	3	4	x
Amount in the account	3500					

How much money will be in the account 15 years later?

2. *Nathan borrows \$35000 for tuition at an interest rate of 5.4% per year, compounded yearly. Complete the table below.*

# of years later	0	1	2	3	x
Amount he owes					

He is able to pay back all the money he borrowed plus the interest soon after graduating, which is 5 years later. How much does he pay back?

3. *You put* \$$P$ *into a savings account that pays interest rate of* r *per year, compounded yearly. Complete the table below that shows how the amount money in your account grows as the years go by.*

# of years later	0	1	2	3	x
Amount in account	P	$P(1+r)$			

4. *On January 1, 2005, Jane deposited \$4000 into a savings account that pays interest at the rate of 4.5% per year, compounded annually. On January 2007 she deposited another \$2000 into the account. She did not deposit into nor withdraw any money from the account until December 2011 when she withdrew all the money and closed the account. How much was in the account when she closed it?*

5. *What annual interest rate will grow \$2500 into \$4600 in 9 years? The interest is compounded every year.*

6. *A certain Credit Union offers savings accounts that pay interest at the rate of 6.4% per year, with the interest credited at the end of every 12 months, provided one does not withdraw any money from the account for 6 years. If a withdraw is made, a penalty amounting to 9% of the amount withdrawn is made. For example, if one withdraws \$400 then a penalty of \$36 is made, so that a total of \$436 is assumed to have been withdrawn from the account. Dennis deposited \$7000 into such an account on January 1, 2004. On January 1, 2007 he withdrew \$2500 from the account. If he made no futher withdraws for the next 3 years, how much money was in the account on December 31, 2010?*

7. *James pays \$6000.00 per year for tuition at his college. There is a savings account that pays interest at the rate of* 4% *per year, compounded yearly. James's sponsors want to deposit one lump sum of money in the account which he will use to pay his tuition.*

   (a) *What is the minimum amount, to the nearest cent, they should deposit into the account today so that he can withdraw \$6000 one year later? (The amount is less than \$6000 because the money earns interest.)*

   (b) *What is the minimum amount, to the nearest cent, they should deposit into the account today so that he can withdraw \$6000 one year from now and another \$6000 two years from now?*

Future Value of an Amount of Money

Example 1 generalizes as follows:

Say you borrow $\$P$ *from a lending institution that charges interest at a rate* i *per year, and you are required to start paying back the loan in one lump sum* n *years later. One year after getting the loan, you will be owing*

$$P + iP = P(1+i) \quad \text{dollars}$$

Two years later you will be owing

$$P(1+i) + iP(1+i) = P(1+i)(1+i) = P(1+i)^2 \quad \text{dollars}$$

The pattern should be clear. The amount you owe increases by a factor of $(1+i)$ *every year. Therefore* t *years after getting the loan, you will be owing* $P(1+i)^t$ *dollars.*

The $20,000 loan the student in Example 1 secured grows by a factor of 1.06 every year. Thus 1 year, later the $20000 loan will have "grown" to $20000(1.06) = 21200$ dollars. We call this the *one-year accumulated value of the loan.* Another term for the same is the **one-year future value of the loan**. Naturally, $20000(1.06)^2$ is the *two-year accumulated*, and in general, $20000(1.06)^n$ is the *n-year accumulated value* or the **n-year future value of the loan**. In general, the **n-year future value of** $P at an interest rate i is $P(1+i)^n$.

Present Value of an Amount of Money

In question 7 of the above exercise, you were asked to determine the minimum amount of money to be deposited now in an account that earns 4% interest per year so that James can withdraw $6000 one year from now. Since an amount of money deposited in an account earning interest at a 4% per annum rate increases by a factor $(1+0.04)$ per year, one way to solve this problem is to answer the question:

$$\text{What is } P \text{ if } P(1+0.04) = 6000?$$

The answer to this question is called the **present value of $6000 paid one year from now** (at an interest rate of 4% per year). Likewise, the answer to the question

$$\text{What is } P \text{ if } P(1+0.04)^2 = 6000?$$

is called the present value of $6000 paid two years from now.

In general, at an annual interest rate i, a lump sum of $P paid n years from now has present value $\dfrac{P}{(1+i)^n} = P(1+i)^{-n}$.

Exercise 3

1. *Erin deposits* $3600 *into a savings account.*

   (a) *Assuming she neither withdraws nor makes deposits into the account, and that the money earns compound interest at an annual rate of 5.4%, how much money will be in the account 4 years later?*

   (b) *If she wants to have* $5000 *in the account 4 years later, what annual compound interest rate must be applied?*

2. *Jane would like to have* $2800 *four years from now for a down-payment on a car. What is the minimum amount of money she must deposit now into a savings account that earns compound interest at an annual rate of 6.2%, so that she has the money at the time?*