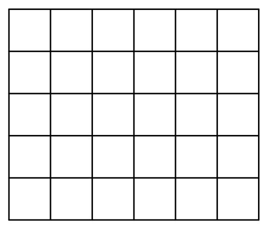
Introduction

Calculus I addressed the problem of calculating slopes of tangents to graphs of functions of one variable and the applications of these slopes to solving a variety of problems. The slopes were also used to define derivatives of functions, hence the term "differential calculus". Calculus II addresses the problem of calculating *limits of sums* and their applications to solving another variety of problems. Among these are: (i) calculating areas enclosed by graphs of functions that need not be straight segments, (ii) calculating volumes of some solids, (iii) calculating work done by variable forces, etc. This chapter presents a problem that may solved using limits of sums. The subsequent chapters introduce the so called *Fundamental Theorem of Calculus* which we use to easily evaluate many such limits.

Calculating some areas

The area of a given region is the number of squares with length 1 unit and width 1 unit that may be fitted into the region. For example, to say that the rectangle in the figure below has area 30 square centimeters means that we can fit thirty squares of length 1 cm and width 1 cm each into the rectangle



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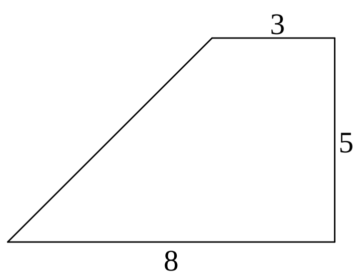
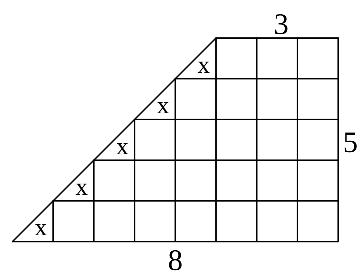
Of course areas do not have to be whole numbers. The area of the trapezium below is $27\frac{1}{2}$



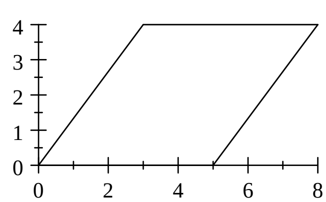
A trapezium



The trapezium sliced into squares

We sliced it into unit squares as shown in the figure to the right. There are 25 full squares plus 5 half squares for a total of $27\frac{1}{2}$ square centimeters.

One may have to rearrange a given figure in order to calculate its area. For example, to figure out the number of unit squares that fit into the parallelogram below with corners at $(0,0)$, $(5,0)$, $(8,4)$ and $(3,4)$, cut out the right triangle ABC



shown in figure (i) and paste it as shown in figure (ii). The result is a rectangle with corners at $(3, 0)$, $(8, 0)$, $(8, 4)$ and $(3, 4)$. A total of 20 unit squares fit into the rectangle, (figure (iii)), therefore the parallelogram has area 20 square units.

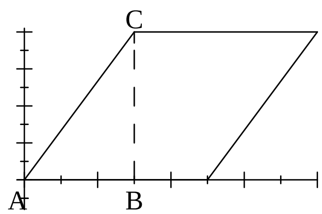


Figure (i)

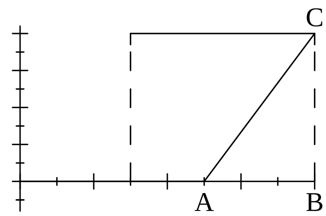


Figure (ii)

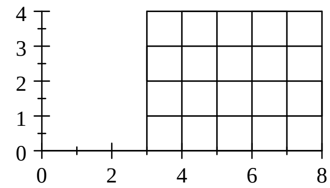
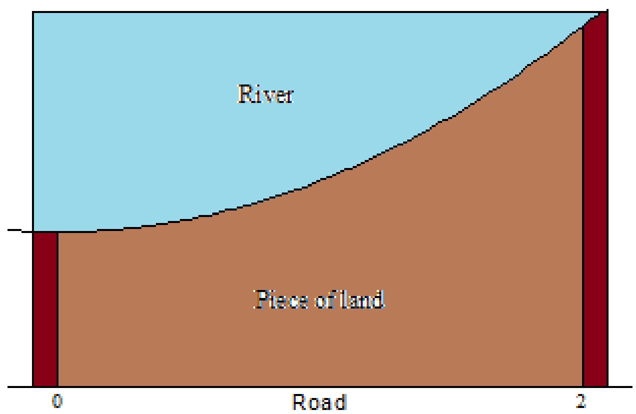
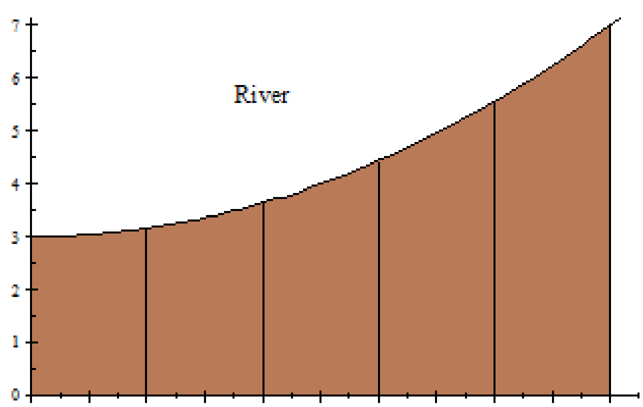


Figure (iii)

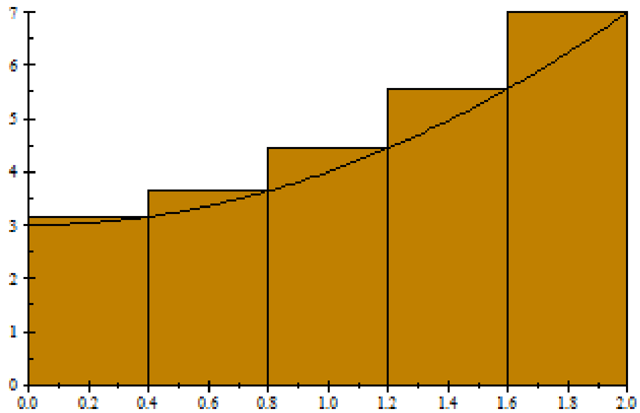
Unfortunately, if a region is enclosed by at least one curve that is not a straight line, then calculating its area requires more than the above calculations. For an example, consider the piece of land shown below, owned by a city. It is between a straight road, which we may imagine to be part of the x-axis, and a river shaped like the graph of the parabola $f(x) = x^2 + 3$, $0 \leq x \leq 2$ where x and $f(x)$ are in miles. Say we are assigned the task of calculating its area as a prelude to determining



its economic value. The complication, of course, is that the river is not a straight line, therefore we have no way of fitting unit squares in some parts of the land. In cases like this, we resort to calculating approximate values of the required area then look for **the single number that is close to all the good approximations.** We get the approximate values by partitioning the piece of land into smaller strips, then approximate each strip with an appropriate rectangle, (since we know how to calculate areas of rectangles). In the figure below, the piece of land is divided into five strips of width 0.4 miles each. Actually, the strips do not have to be of equal width; we chose equal width here to simplify the computations that follow.



Each strip is then approximated by a rectangle. The figure below shows our choice of approximating rectangles. Each rectangle is "bigger" than the strip it approximates. The error in a particular rectangle is the area of the part of the rectangle that is above the parabola.



Approximating rectangles

The area of the left-most rectangle is

$$(0.4)\left(3+0.4^2\right) \text{ square miles.}$$

The next one has area

$$(0.4)\left(3+0.8^2\right) \text{ square miles.}$$

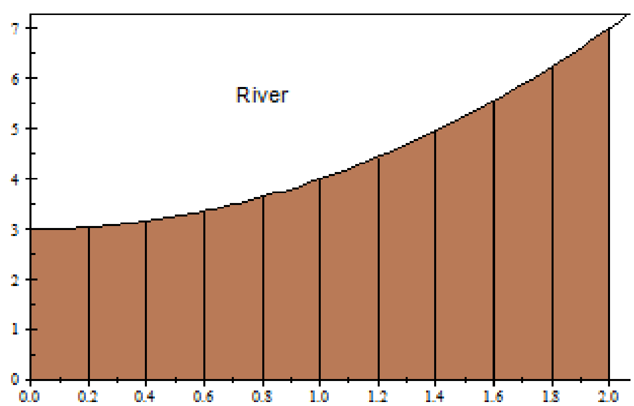
The areas of the other three approximating rectangles are calculated in the same way. The total area of the five approximating rectangles is

$$0.4\left[3+0.4^2+3+0.8^2+3+1.2^2+3+1.6^2+3+2^2\right] \text{ square miles.}$$

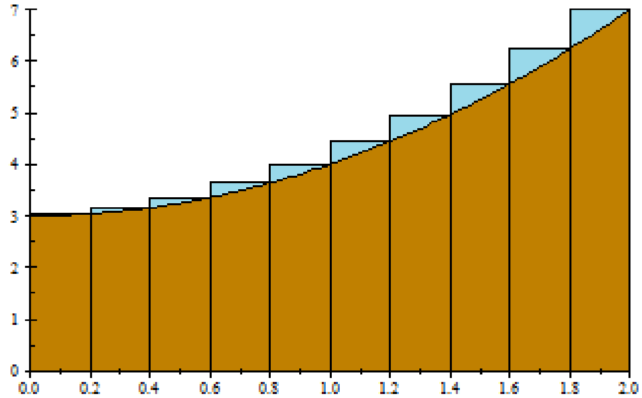
Therefore, if the exact area of the land is A square miles then

$$A \simeq 0.4\left[3+0.4^2+3+0.8^2+3+1.2^2+3+1.6^2+3+2^2\right] = 9.52 \text{ square miles.}$$

The error in this estimate is the total area of the regions that are above the parabola. We can reduce it by approximating the region with smaller rectangles as shown below where we have partitioned the land into 10 smaller strips.



Ten strips

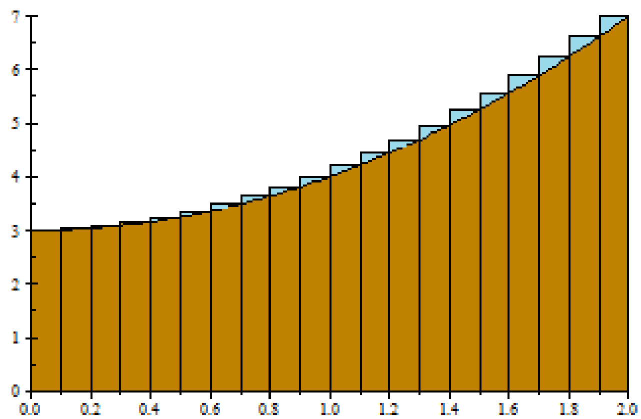


Corresponding approximating rectangles

The error is definitely reduced because part of the region included in the first estimate is not included in the smaller rectangles. Therefore, a better approximation is

$$A \simeq 0.2\left[3+0.2^2+3+0.4^2+3+0.6^2+\cdots+3+1.8^2+3+2^2\right]=9.08$$

An even better approximation is obtained by partitioning the region into 20 small strips as shown below



Its value is

$$A \simeq 0.1\left[3+0.1^2+3+0.2^2+3+0.3^2+\cdots+3+1.9^2+3+2^2\right]=8.87$$

We now take a general step: Imagine dividing the region into n strips of width $\frac{2}{n}$ each then approximate each strip with a rectangle as above. We obtain the following estimate:

$$A \simeq \frac{2}{n}\left[3+\left(\frac{2}{n}\right)^2+3+\left(\frac{4}{n}\right)^2+3+\left(\frac{6}{n}\right)^2+\cdots+3+\left(\frac{2n-2}{n}\right)^2+3+\left(\frac{2n}{n}\right)^2\right]$$

This sum includes n three's. They add up to $3n$. Also, note that

$$\left(\frac{2}{n}\right)^2=\frac{4}{n^2}, \qquad \left(\frac{4}{n}\right)^2=\frac{4}{n^2}(2)^2, \qquad \left(\frac{6}{n}\right)^2=\frac{4}{n^2}\left(3^2\right), \qquad \ldots, \qquad \left(\frac{2n}{n}\right)^2=\frac{4}{n^2}(n)^2$$

For convenience, we write $\frac{4}{n^2}$ as $\frac{4}{n^2}(1)^2$. Then

$$\begin{aligned} A &\simeq \frac{2}{n}\left[3n+\frac{4}{n^2}(1)^2+\frac{4}{n^2}(2)^2+\frac{4}{n^2}(3)^2+\cdots+\frac{4}{n^2}(n)^2\right] \\ &= 6+\frac{8}{n^3}\left[1^2+2^2+3^2+\cdots+n^2\right] \end{aligned}$$

There is a formula for the sum of the squares of the first n positive integers. It is derived in Chapter **??** and it is

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Applying it to the above sum gives

$$\begin{aligned} A &\simeq 6 + \frac{8}{n^3}\left(\tfrac{n(n+1)(2n+1)}{6}\right) = 6 + \tfrac{4}{3}\left(\tfrac{n+1}{n}\right)\left(\tfrac{2n+1}{n}\right) \\ &= 6 + \tfrac{4}{3}\left(1 + \tfrac{1}{n}\right)\left(2 + \tfrac{1}{n}\right) \end{aligned}$$

Using this formula gives the following table:

# of rectangles	Area of approximating rectangles
10	$6 + \frac{4}{3}\left(1 + \frac{1}{10}\right)\left(2 + \frac{1}{10}\right) = 9.08$
20	$6 + \frac{4}{3}\left(1 + \frac{1}{20}\right)\left(2 + \frac{1}{20}\right) = 8.87$
40	$6 + \frac{4}{3}\left(1 + \frac{1}{40}\right)\left(2 + \frac{1}{40}\right) = 8.7675$
100	$6 + \frac{4}{3}\left(1 + \frac{1}{100}\right)\left(2 + \frac{1}{100}\right) = 8.7068$
1000	$6 + \frac{4}{3}\left(1 + \frac{1}{1000}\right)\left(2 + \frac{1}{1000}\right) = 8.670668$
40000	$6 + \frac{4}{3}\left(1 + \frac{1}{40000}\right)\left(2 + \frac{1}{40000}\right) = 8.66677$ (to5 dec. pl.)
2×10^6	$6 + \frac{4}{3}\left(1 + \frac{1}{2\times 10^6}\right)\left(2 + \frac{1}{2\times 10^6}\right) = 8.6666687$ (to7 dec. pl.)

The table and the fact that the value of $6 + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$ is close to $6 + \frac{8}{3} = \frac{26}{3}$ for all large integers n suggest that $\frac{26}{3}$ is the single number that is close to all the "*good approximations*" of A. Therefore, $\frac{26}{3}$ is the best candidate for the required area of the piece of land.