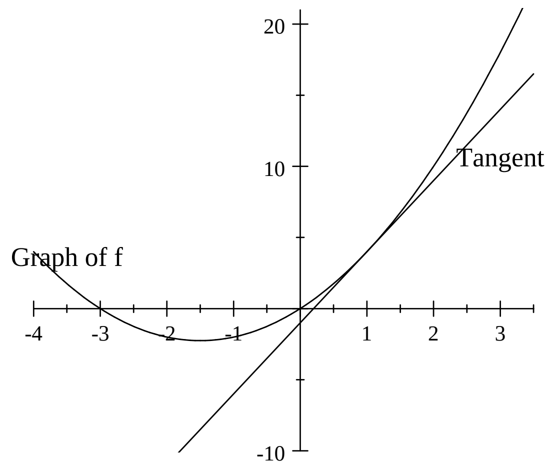
Introduction

Calculus is a tool we use to determine values of various quantities that we may not be able to obtain by performing a finite number of the familiar addition/multiplication operations. The calculus approach to such a problem is to calculate approximate values of the required quantity then ask the following question: "**What single number is close to all the good approximations**?" That number, if it exists, is called the limit of the approximations and it is the best candidate for the exact value of the required quantity. The following example sheds some light on this approach:

Example 1 *To determine the slope* m *of the tangent to the graph of* $f(x) = x^2 + 3x$ *at* $(1, 4)$

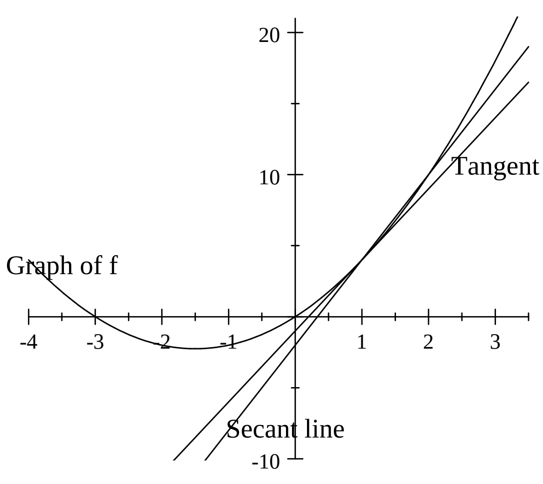


The graph of f *and its tangent at* $(1, 4)$ *are shown in the figure above. Unfortunately, we know only one point, namely* $(1, 4)$*, on the tangent therefore we do not have enough information to determine its slope using the formula*

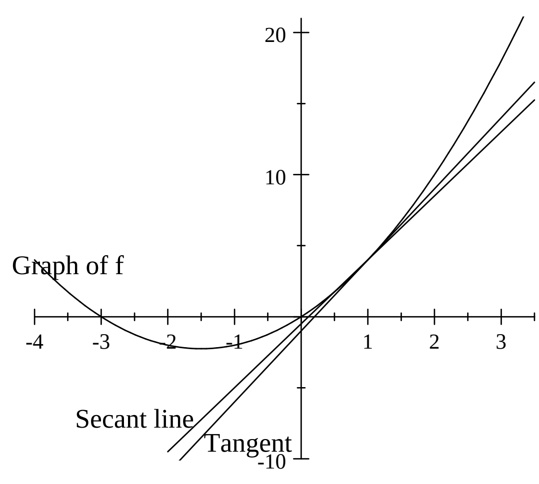
$$m = \frac{Rise}{Run}$$

(We need two points on a given line in order to calculate the Rise and Run.)

The calculus approach is to calculate approximate values of m *and figure out the exact value from the approximations. An approximation may be obtained by approximating the tangent with a line segments, called secant line, that pass through* $(1, 4)$ *and one other point on the graph of* f*. We can calculate the slope of a secant line because we have two of its points. The figures below show the secant line through* $(1, 4)$ *and* $(2, 10)$ *and the secant line through* $(1, 4)$ *and* $(0.5, 1.75)$*.*



Secant line through $(1, 4)$ *and* $(2, 10)$



Secant line through $(1, 4)$ *and* $(0.5, 1.75)$

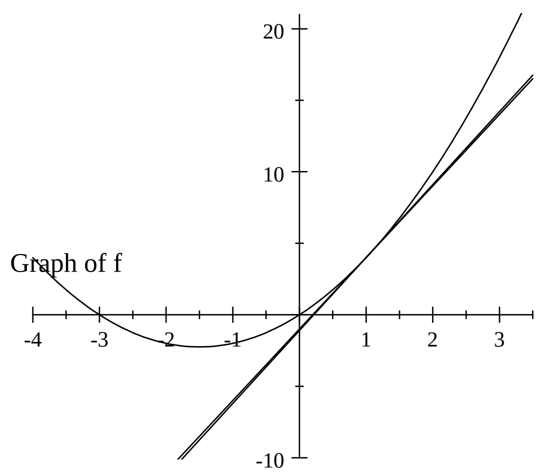
The slope of the first one is $\dfrac{10-4}{2-1} = 6$, *therefore* $m \simeq 6$. *Of course this is just one of the many approximations of* m. *The secant line through* $(1,4)$ *and* $(0.5, 1.75)$ *provides another approximation which is* $m \simeq \dfrac{1.75-4}{0.5-1} = 4.5$. *Since, as the figure to the right shows, the secant line through* $(1,4)$ *and* $(0.5, 1.75)$ *is "closer" to the tangent than the secant line through* $(1,4)$ *and* $(2,10)$, *the number* 4.5 *is closer to* m *than the number 6.*

Rather than evaluating approximate values by picking arbitrary points, it is more efficient to evaluate the slope of a general secant line and use it to determine specific ones. To this end, take a general point on the graph of f *that is close to* $(1,4)$. *It has the general form* $\left(1+h, (1+h)^2 + 3(1+h)\right)$ *where* h *is a small number which may be positive or negative number. For example, if we choose* $h = 1$ *we get the point* $(2,10)$ *that we used to get* $m \simeq 6$. *The other point* $(0.5, 1.75)$ *is obtained by choosing* $h = -0.5$.

The slope of the secant line joining $(1,4)$ *and* $\left(1+h, (1+h)^2 + 3(1+h)\right)$ *is*

$$\frac{(1+h)^2 + 3(1+h) - 4}{(1+h) - 1} = \frac{1 + 2h + h^2 + 3h + 3 - 4}{h} = \frac{5h + h^2}{h} = \frac{h(5+h)}{h} = 5 + h$$

Now it is a matter of choosing different values of h *to get approximate values of* m. *In particular, choosing* $h = 1$ *gives the approximation* $5 + 1 = 6$ *and choosing* $h = -0.5$ *gives the approximation* 4.5 *which we obtained above. We note that the good approximations to the tangent are obtained by choosing small values of* h. *The figure below shows the tangent and the approximating secant line when* $h = 0.1$. *The tangent and secant line are almost indistinguishable.*



It stands to reason that the good approximate values of m *are the numbers of the form* $5 + h$ *where* h *is a tiny number. The following table gives sample values*

Value of h	*Secant line through*	*Approximate value of* m
0.1	$(1,4)$ *and* $(1.1, 4.51)$	5.1
0.02	$(1,4)$ *and* $(1.02, 4.1004)$	5.02
-0.01	$(1,4)$ *and* $(0.99, 3.9501)$	4.99
0.001	$(1,4)$ *and* $(1.001, 4.005001)$	5.001
-0.0001	$(1,4)$ *and* $(0.99999, 3.99950001)$	4.9999

It should be clear from this table and the general expression $5 + h$ *for the slope of the secant line through* $(1,4)$ *and* $\left(1+h, (1+h)^2 + 3(1+h)\right)$ *that the single number that* ***is close to all the good approximate values of*** m ***is*** 5. *This must be the slope of the tangent. Later in the course, we will state more precisely the phrase in bold letters.*

Exercise 2 *For practice, use the same approach to determine the slope of the tangent to the graph of* $g(x) = 2x^2 + x$ *at the point* $(-2, 6)$.