Math 154 review for Final Examination

1. Complete the table

Function	Derivative
$\left(3x^2+5\right)^8$	$8\left(3x^2+5\right)^7(3x)=24x\left(3x^2+5\right)^7$
$\sqrt{x-\dfrac{1}{x}}$	$\frac{1}{2}\left(x-\dfrac{1}{x}\right)^{-\frac{1}{2}}\left(1+\dfrac{1}{x^2}\right)$
$\dfrac{e^{2x}}{x}=x^{-1}e^{2x}$	$-x^{-2}e^{2x}+e^{2x}(2)\left(x^{-1}\right)=\left(\dfrac{2}{x}-\dfrac{1}{x^2}\right)e^{2x}$
$3x\sqrt{x^2+5}$	$3\left(x^2+5\right)^{\frac{1}{2}}+\frac{1}{2}\left(x^2+5\right)^{-\frac{1}{2}}(2x)(3x)=3\left(x^2+5\right)^{\frac{1}{2}}+3x^2\left(x^2+5\right)^{-\frac{1}{2}}$
$(4x+1)^{3/2}$	$\frac{3}{2}(4x+1)^{\frac{1}{2}}(4)=6(4x+1)^{\frac{1}{2}}$
$4x-\frac{1}{5}\cos 5x+c$	$4+\sin 5x$
$3x-2\sin\frac{1}{2}x+c$	$3-\cos\frac{1}{2}x$
$[\ln(4x+7)]^{3/4}$	$\frac{3}{4}[\ln(4x+7)]^{-\frac{1}{4}}\left(\dfrac{4}{4x+7}\right)=\left(\dfrac{3}{4x+7}\right)[\ln(4x+7)]^{-\frac{1}{4}}$
$\sec^2 4x+\tan 3x$	$2(\sec 4x)(\sec 4x\tan 4x)4+\left(\sec^2 3x\right)3=8\sec^2 4x\tan 4x+3\sec^2(3x)$
x^3e^{3x}	$3x^2e^{3x}+\left(e^{3x}\right)(3)x^3=3\left(x^2+x^3\right)e^{3x}$
$4\sin x\cos 2x\tan 3x$	$4\cos x\cos 2x\tan 3x-8\sin x\sin 2x\tan 3x+12\sin x\cos 2x\sec^2 3x$

2. Evaluate each definite integral:

a) $\displaystyle\int_0^2\left(x^3-3x^2+6x-1\right)dx=\left[\tfrac{x^4}{4}-x^3+3x^2-x\right]_0^2=(4-8+12-2)-(0)=6$

b) $$\int_1^4\left(\frac{2}{x}-\frac{4}{3x^2}+6x^2-1\right)dx=\left[2\ln x+\frac{4}{3x}+2x^3-x\right]_1^4=2\ln 4+\tfrac{1}{3}+128-4-\left(0+\tfrac{4}{3}+2-1\right)$$
$$=2\ln 4-\tfrac{3}{3}+124-1=2\ln 4+122=\ln 16+122$$

c) $$\int_0^1\left(\frac{x^2}{x^3+5}-6e^x\right)dx=\left[\frac{1}{3}\ln\left(x^3+5\right)-6e^x\right]_0^1=\frac{1}{3}\ln 6-6e-\left(\tfrac{1}{3}\ln 5-6\right)$$
$$=\frac{1}{3}\ln\tfrac{6}{5}-6e+6$$

d) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(3 - \sec^2 x - 2\csc^2 x\right) dx = \Big[3x - \tan x + 2\cot x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \pi - \sqrt{3} + \tfrac{2}{\sqrt{3}} - \left(\tfrac{\pi}{2} - \tfrac{1}{\sqrt{3}} + 2\sqrt{3}\right)$$

$$= \frac{1}{3}\ln\tfrac{6}{5} - 6e + 6$$

e) $$\int_0^{\pi/4} (2 + \sec x \tan x) dx = \Big[2x - \sec x\Big]_0^{\frac{\pi}{4}} = \tfrac{\pi}{2} - \sqrt{2} - (0 - 1) = \tfrac{\pi}{2} + 1 - \sqrt{2}$$

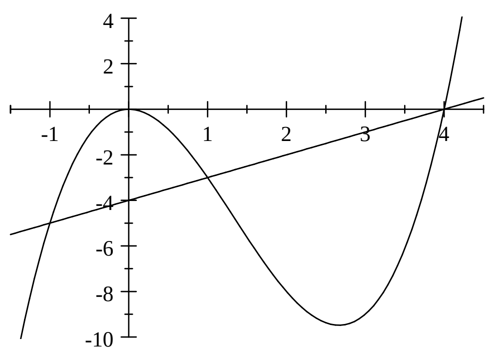
f) $$\int_0^1 \left(\frac{3e^{2x}}{4} - \frac{1}{1+x^2}\right) dx = \left[\frac{3e^{2x}}{8} - \arctan x\right]_0^1 = \frac{3e}{8} - \arctan 1 - \left(\frac{3}{8} - 0\right) = \frac{3}{8}(e-1) - \tfrac{\pi}{4}$$

g) $$\int_{-\pi/6}^{\pi/4} (\sin 2x + 3\cos 2x - 1)\, dx = \left[-\frac{\cos 2x}{2} + \frac{3\sin 2x}{2} - x\right]_{\frac{-\pi}{6}}^{\frac{\pi}{4}} = 0 + \frac{3}{2} - \tfrac{\pi}{4} - \left(\frac{1}{4} - \frac{3\sqrt{3}}{2} + \frac{\pi}{6}\right)$$

$$\frac{3}{2} + \frac{1}{4} + \frac{3\sqrt{3}}{2} - \frac{\pi}{4} - \frac{\pi}{6} = \frac{21 + 18\sqrt{3} - 5\pi}{12}$$

h) $$\int_{-1}^{\frac{1}{2}} \left(8 - \frac{3}{\sqrt{1-x^2}}\right) dx = \Big[8x - 3\arcsin x\Big]_{-1}^{\frac{1}{2}} = 4 - 3\left(\tfrac{\pi}{6}\right) - \left(-8 + 3\left(\tfrac{\pi}{2}\right)\right) = 12 - 2\pi$$

3. Let $f(x) = x^3 - 4x^2$ and $g(x) = x - 4$. Their graphs are given below



(a) Show that the graphs of f and g intersect at points where $x = -1$, $x = 1$ and $x = 4$.

Solution: Where the two graphs intersect, $x^3 - 4x^2 = x - 4$. The left hand side of the equation factors as $x^2(x-4)$. Therefore

$x^2(x-4) = (x-4)$ I.e. $x^2(x-4) - (x-4) = 0$, which is equivalent to $\left(x^2 - 1\right)(x-4) = 0$

This may be factored as $(x-1)(x+1)(x-4) = 0$. Therefore, where they intersect, $x = -1$ or $x = 1$ or $x = 4$

(b) Calculate the area enclosed by the two graphs.

Solution: On the interval $[-1, 1]$, the graph of f is above the graph of g, therefore the area they enclose on this interval is

$$\int_{-1}^1 [f(x) - g(x)]\, dx = \int_{-1}^1 \left[x^3 - 4x^2 - (x-4)\right] dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 - \frac{x^2}{2} + 4x\right]_{-1}^1$$

$$= \frac{1}{4} - \frac{4}{3} - \frac{1}{2} + 4 - \left(\frac{1}{4} + \frac{4}{3} - \frac{1}{2} - 4\right) = 8 - \frac{4}{3} = \frac{20}{3}$$

On the interval $[1,4]$, the graph of g is above the graph of f, therefore the area they enclose on this interval is

$$\begin{aligned}\int_{-1}^{1}[g(x)-f(x)]\,dx &= \int_{-1}^{1}\left[x-4-\left(x^3-4x^2\right)\right]dx = \left[\frac{x^2}{2}-4x-\frac{x^4}{4}+\frac{4}{3}x^3\right]_1^4 \\ &= 8-16-64+\frac{256}{3}-\left(\frac{1}{2}-4-\frac{1}{4}+\frac{4}{3}\right)=\frac{63}{4}\end{aligned}$$

4. Show that substituting $u=1+x^2$ in the definite integral $\int x^5\sqrt{1+x^2}dx$ gives

$$\frac{1}{2}\int(u-1)^2\sqrt{u}du=\frac{1}{2}\int\left(u^{5/2}-2u^{3/2}+u^{1/2}\right)du$$

(a) **Solution**: Let $u=1+x^2$. Then $\frac{du}{dx}=2x$, therefore $dx=\frac{du}{2x}$. It follows that

$$x^5\sqrt{1+x^2}dx=x^5\sqrt{u}\frac{du}{2x}=x^4\sqrt{u}\frac{du}{2}$$

To replace x in the expression $x^4\sqrt{u}\frac{du}{2}$, use the substitution equation $u=1+x^2$ determine x^4 in terms of. Since $u=1+x^2$, it follows that $x^2=(u-1)$, therefore $x^4=(u-1)^2=u^2-2u+1$. We get $x^5\sqrt{1+x^2}dx=\left(u^2-2u+1\right)\sqrt{u}du$ which implies that

$$\int x^5\sqrt{1+x^2}dx=\int\left(u^2-2u+1\right)\sqrt{u}du=\int\left(u^{5/2}-2u^{3/2}+^{1/2}u\right)du$$

To evaluate the definite integral $\int_0^{\sqrt{3}}x^5\sqrt{1+x^2}dx$, use the above substitution. When $x=0$, $u=1$ and when $x=\sqrt{3}$, $u=4$. Therefore

$$\begin{aligned}\int_0^{\sqrt{3}}x^5\sqrt{1+x^2}dx &= \int_1^4\left(u^{5/2}-2u^{3/2}+^{1/2}u\right)du=\left[\frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right]_1^4 \\ &= \frac{2}{7}(128)-\frac{4}{5}(32)+\frac{2}{3}(8)-\left(\frac{2}{7}-\frac{4}{5}+\frac{2}{3}\right)=16.15\text{, to 2 dec. pl.}\end{aligned}$$

5. You have to evaluate the definite integral $\int_0^{\pi/3}x^2\sin 2xdx$

(a) Use integration by parts to show that $\int x^2\sin 2xdx=-\frac{x^2\cos 2x}{2}+\int x\cos 2xdx$.

Solution: Using the integration by parts formula in the form $\int f(x)g'(x)dx=f(x)g(x)-\int g'(x)f(x)dx$ to $\int x^2\sin 2xdx$, take $f(x)=x^2$ and $g'(x)=\sin 2x$. Then $f'(x)=2x$ and $g(x)=-\frac{\cos 2x}{2}$. Therefore

$$\int x^2\sin 2xdx=\left(x^2\right)\left(-\frac{\cos 2x}{2}\right)-\int\left(-\frac{\cos 2x}{2}\right)[2x]\,dx=-\frac{x^2\cos 2x}{2}+\int x\cos 2xdx.$$

Applying the formula to $\int x\cos 2xdx$, take $f(x)=x$ and $g'(x)=\cos 2x$. Then $f'(x)=1$ and $g(x)=\frac{\sin 2x}{2}$. Therefore

$$\int x\cos 2xdx=$$

(b) Use another integration by parts to show that $\int x\cos 2x dx = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}$.

Solution: To apply the formula to $\int x\cos 2x dx$, take $f(x) = x$ and $g'(x) = \cos 2x$. Then $f'(x) = 1$ and $g(x) = \frac{\sin 2x}{2}$. Therefore

$$\int x\cos 2x dx = (x)\left(\frac{\sin 2x}{2}\right) - \int \left(\frac{\sin 2x}{2}\right)(1)\,dx = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2} = \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

It follows that

$$\int x^2\sin 2x dx = -\frac{x^2\cos 2x}{2} + \int x\cos 2x dx = -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + c$$

(c) Finally, evaluate $\int_0^{\pi/3} x^2\sin 2x dx$

Solution:

$$\begin{aligned}\int_0^{\pi/3} x^2\sin 2x dx &= \left[-\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{\pi/3} \\ &= \frac{\pi^2}{36} + \frac{\sqrt{3}\pi}{12} - \frac{1}{8} - \left(0 + 0 + \tfrac{1}{4}\right) = \frac{2\pi^2 + 6\sqrt{3}\pi - 27}{72}\end{aligned}$$

6. You are given the function $f(x) = \frac{2}{x(x+2)}$.

(a) Split $\frac{2}{x(x+2)}$ into partial fractions then show that

$$\int \frac{2}{x(x+2)} = \ln\left|\frac{x}{x+2}\right| + c$$

Solution: $\frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{(x+2)}$. Multiply each term by $x(x+2)$ to clear fractions. The result is

$$2 = A(x+2) + Bx$$

Substituting $x = 0$ gives $2 = 2A$, therefore $A = 1$. Substituting $x = -2$ gives $2 = -2B$, therefore $B = -1$. The conclusion is that $\frac{2}{x(x+2)} = \frac{1}{x} - \frac{1}{(x+2)}$, therefore

$$\int \frac{2}{x(x+2)} = \int \frac{1}{x} - \int \frac{1}{(x+2)} = \ln|x| - \ln|x+2| + c = \ln\left|\frac{x}{x+2}\right| + c$$

(b) Now show that $\int_1^{\infty} \frac{2dx}{x(x+2)} = \ln 3$

Solution: By definition, $\int_1^{\infty} \frac{2dx}{x(x+2)}$ is the limit of $\int_1^{R} \frac{2dx}{x(x+2)}$ as R approaches ∞. Therefore

$$\begin{aligned}\int_1^{\infty} \frac{2dx}{x(x+2)} &= \lim_{R\to\infty}\int_1^{R} \frac{2dx}{x(x+2)} = \lim_{R\to\infty}\left[\ln\left|\frac{x}{x+2}\right|\right]_1^R = \lim_{R\to\infty}\left(\ln\left|\frac{R}{R+2}\right| - \ln\left|\frac{1}{3}\right|\right) \\ &= \ln 1 - \ln\left(\frac{1}{3}\right) = 0 - \ln\left(\frac{1}{3}\right) = \ln 3\end{aligned}$$

7. You are required to evaluate the definite integral $\int_0^2 \frac{x^2}{\sqrt{16-x^2}}dx$

(a) Show that substituting $x = 4\sin\theta$ into $\int_0^2 \frac{x^2}{\sqrt{16-x^2}}dx$ gives $16\int_0^{\pi/6} \sin^2\theta d\theta$.

Solution: Let $x = 4\sin\theta$. Then $\frac{dx}{d\theta} = 4\cos\theta$ and so $dx = 4\cos\theta d\theta$. Therefore

$$\frac{x^2dx}{\sqrt{16-x^2}} = \frac{\left(16\sin^2\theta\right)\left(4\cos\theta d\theta\right)}{\sqrt{16-16\sin^2\theta}} = \frac{\left(64\sin^2\theta\right)\left(\cos\theta d\theta\right)}{\sqrt{16(1-\sin^2\theta)}} = \frac{\left(64\sin^2\theta\right)\left(\cos\theta d\theta\right)}{\sqrt{16\cos^2\theta)}}$$

$$= \frac{\left(64\sin^2\theta\right)\left(4\cos\theta d\theta\right)}{4\cos\theta} = 16\sin^2\theta d\theta$$

When $x = 0$, then $0 = 4\sin\theta$. This implies that $\sin\theta = 0$ and so $\theta = 0$. When $x = 2$, then $2 = 4\sin\theta$. This implies that $\sin\theta = \frac{1}{2}$ and so $\theta = \frac{\pi}{6}$. Now the integral becomes

$$\int_0^2 \frac{x^2}{\sqrt{16-x^2}}dx = \int_0^{\pi/6} 16\sin^2\theta d\theta = 16\int_0^{\pi/6} \sin^2\theta d\theta$$

(b) Now show that $\int_0^2 \frac{x^2}{\sqrt{16-x^2}}dx = \frac{4\pi}{3} - 2\sqrt{3}$.

Solution:

$$\int_0^2 \frac{x^2}{\sqrt{16-x^2}}dx = 16\int_0^{\pi/6} \sin^2\theta d\theta = 8\int_0^{\pi/6}(1-\cos 2\theta)d\theta = 8\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$$

$$= 8\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - 8\left(0-0\right) = \frac{4\pi}{3} - 2\sqrt{3}.$$

8. Show that substituting $u = 3x+1$ into $\int \frac{x^2}{\sqrt{3x+1}}dx$ gives $\frac{1}{27}\int \frac{(u-1)^2}{\sqrt{u}}du$ then evaluate the definite integral

$$\int_0^5 \frac{x^2}{\sqrt{3x+1}}dx.$$

(a) **Solution**: Let $u = 3x+1$. Then $\frac{du}{dx} = 3$ and so $dx = \frac{du}{3}$. Solving for x using the substitution equation $3x+1 = u$ gives $x = \frac{u-1}{3}$. It follows that

$$\frac{x^2}{\sqrt{3x+1}}dx = \frac{\left(\frac{u-1}{3}\right)^2}{\sqrt{u}}\left(\frac{du}{3}\right) = \frac{(u-1)^3}{27\sqrt{u}}du$$

This implies that

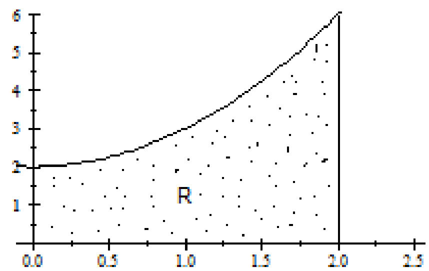
$$\int \frac{x^2}{\sqrt{3x+1}}dx = \frac{1}{27}\int \frac{(u-1)^2}{\sqrt{u}}du = \frac{1}{27}\int \frac{\left(u^2-2u+1\right)}{\sqrt{u}}du = \frac{1}{27}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$$

When $x = 0$, then $u = 1$ and when $x = 5$, then $u = 16$. Therefore the definite integral becomes

$$\int_0^5 \frac{x^2}{\sqrt{3x+1}}dx = \frac{1}{27}\int_1^{16}\left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du = \frac{1}{27}\left[\frac{2u^{\frac{5}{2}}}{5} - \frac{4u^{\frac{3}{2}}}{3} + 2u^{\frac{1}{2}}\right]_1^{16}$$

$$= \frac{1}{27}\left(\frac{1024}{5} - \frac{256}{3} + 8\right) - \frac{1}{27}\left(\frac{2}{5} - \frac{4}{3} + 2\right) = \frac{632}{405}$$

9. The figure below shows the region R enclosed by the graph of $f(x) = x^2 + 2$, the x-axis, the y-axis and the line $x = 2$. Sketch, to the right of the figure, the solid obtained by rotating R about the x-axis through four right angles, then calculate its volume.



(a) **Solution**: The volume V of the solid is given by the formula

$$\begin{aligned} V &= \pi \int_0^2 \left(x^2+2\right)^2 dx = \pi \int_0^2 \left(x^4+4x^2+4\right)^2 dx \\ &= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right]_0^2 = \frac{376\pi}{15} \text{ cubic units} \end{aligned}$$

10. Let $f(x) = \frac{2}{3}x^{3/2}$. Calculate the length of the graph of f between $x = 0$ and $x = 8$.

(a) **Solution**: The length L of the graph is given by the formula

$$\begin{aligned} L &= \int_0^8 \sqrt{1+\left(f'(x)\right)^2} dx = \int_0^8 \sqrt{1+\left(x^{\frac{1}{2}}\right)^2} dx = \int_0^8 \sqrt{1+x} dx \\ &= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^8 = \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} = \frac{52}{3} \text{ units} \end{aligned}$$

11. Show that substituting $x = \tan u$ into $\int \frac{1}{\left(1+x^2\right)^2} dx$ gives $\int \frac{du}{\sec^2 u} = \int \cos^2 u du$ then evaluate the definite integral

$$\int_0^1 \frac{1}{\left(1+x^2\right)^2} dx.$$

(a) **Solution**: Let $x = \tan u$. Then $\frac{dx}{du} = \sec^2 u$ and so $dx = \sec^2 u du$ and

$$\frac{1}{\left(1+x^2\right)^2} dx = \frac{\sec^2 u du}{\left(1+\tan^2 u\right)^2} = \frac{\sec^2 u du}{\left(\sec^2 u\right)^2} = \frac{\sec^2 u du}{\sec^4 u} = \frac{du}{\sec^2 u} = \cos^2 u du$$

Thus the substitution changes the integral into $\int \cos^2 u du$. When $x = 0$, then $\tan u = 0$ and so $u = 0$. When $x = 1$ then $\tan u = 1$ and so $u = \frac{\pi}{4}$. Therefore the definite integral $\int_0^1 \frac{1}{\left(1+x^2\right)^2} dx$ is transformed into

$$\int_0^1 \frac{1}{\left(1+x^2\right)^2} dx = \int_0^{\frac{\pi}{4}} \cos^2 u du = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1+\cos 2u)\, du = \left[\frac{1}{2}\left(u + \frac{\sin 2u}{2}\right)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{4}$$

12. You are given the definite integral $\int_0^2 e^{-\frac{1}{2}x^2} dx$. (This was a quiz problem.)

(a) Estimate it using the trapezoidal rule with $n = 8$.

(b) Estimate it using Simpson's rule with $n = 8$.

(c) Estimate it using the midpoint rule, with $n = 8$.

13. Let $f(x) = 2\sqrt{x}$.

(a) Show that $\left(\sqrt{1+[f'(x)]^2}\right) f(x) = 2\sqrt{x+1}$.

Solution: Since $f(x) = 2x^{\frac{1}{2}}$, it follows that $f'(x) = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{\sqrt{x}}$, therefore

$$\begin{aligned}\left(\sqrt{1+[f'(x)]^2}\right) f(x) &= \left(\sqrt{\left(1+\frac{1}{x}\right)}\right) 2\sqrt{x} = \left(2\sqrt{x}\right)\sqrt{\frac{x+1}{x}} \\ &= \left(2\sqrt{x}\right)\frac{\sqrt{x+1}}{\sqrt{x}} = 2\sqrt{x+1}\end{aligned}$$

(b) Determine the area of the surface generated when the graph of f between $x = 0$ and $x = 3$ is rotated about the x - axis through 4 right angles.

Solution: The area A of the surface generated is given by the formula

$$\begin{aligned}A &= \int_0^3 \left(\sqrt{1+[f'(x)]^2}\right) f(x)dx = \int_0^3 2\sqrt{x+1}dx = \left[\frac{4}{3}(x+1)^{\frac{3}{2}}\right]_0^3 \\ &= \frac{32}{3} - \frac{4}{3} = \frac{28}{3} \quad \text{square units.}\end{aligned}$$

14. You are given the rational function $f(x) = \dfrac{1}{(x+1)(2x+3)}$.

(a) Split it into partial fractions then show that $\dfrac{A}{(x+1)} + \dfrac{B}{(2x+3)} = \ln\left|\dfrac{x+1}{2x+3}\right| + c$.

Solution: $\dfrac{1}{(x+1)(2x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(2x+3)}$. To clear fractions, multiply every term by $(x+1)(2x+3)$. The result is

$$1 = A(2x+3) + B(x+1)$$

Substituting $x = -1$ gives $1 = A(-2+3) = A$, therefore $A = 1$. Substituting $x = -\frac{3}{2}$ gives $1 = B(-\frac{1}{2})$, therefore $B = -2$. It follows that

$$\frac{1}{(x+1)(2x+3)} = \frac{1}{(x+1)} - \frac{2}{(2x+3)}$$

hence $\displaystyle\int \frac{1}{(x+1)(2x+3)} = \int \frac{1}{(x+1)} - \int \frac{2}{(2x+3)}$. These are both log integrals because the derivative of each denominator is the numerator. It follows that

$$\begin{aligned}\int \frac{1}{(x+1)(2x+3)} &= \int \frac{1}{(x+1)} - \int \frac{2}{(2x+3)} \\ &= \ln|x+1| - \ln|2x+3| + c = \ln\left|\frac{x+1}{2x+3}\right| + c.\end{aligned}$$

(b) Now show that $\displaystyle\int_0^\infty \frac{dx}{(x+1)(2x+3)} = \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right) = \ln\left(\frac{3}{2}\right)$.

Solution: By definition, $\int_0^{\infty} \frac{dx}{(x+1)(2x+3)}$ is the limit of $\int_0^{R} \frac{dx}{(x+1)(2x+3)}$ as R approaches ∞. Thus

$$\begin{aligned}\int_0^{\infty} \frac{dx}{(x+1)(2x+3)} &= \lim_{R\to\infty} \int_0^{R} \left(\frac{1}{(x+1)} - \frac{2}{(2x+3)} \right) dx = \lim_{R\to\infty} \left[\ln \left| \frac{x+1}{2x+3} \right| \right]_0^R \\ &= \lim_{R\to\infty} \left(\ln \left| \frac{R+1}{2R+3} \right| - \ln \left| \frac{1}{3} \right| \right) = \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right) = \ln\left(\frac{3}{2}\right)\end{aligned}$$

15. Use the substitution $u = 1 + 3x^2$ to evaluate the definite integral $\int_0^1 x^3\sqrt{1+3x^2}dx$.

(a) **Solution**: Let $u = 1 + 3x^2$. Then $\frac{du}{dx} = 6x$ and so $dx = \frac{du}{6x}$. It follows that

$$x^3\sqrt{1+3x^2}dx = \left(x^3\sqrt{u}\right)\frac{du}{6x} = x^2\frac{\sqrt{u}}{6}du$$

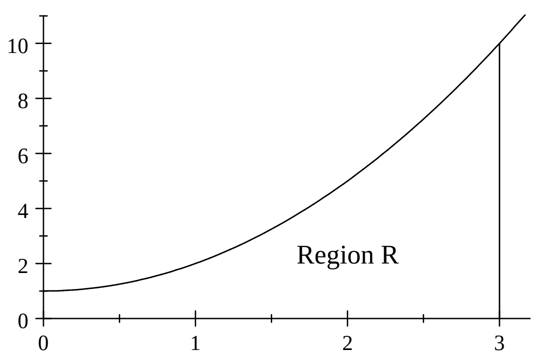
To replace x^2, we use the substitution formula $u = 1 + 3x^2$. The result is $x^2 = \left(\frac{u-1}{3}\right)$. Therefore

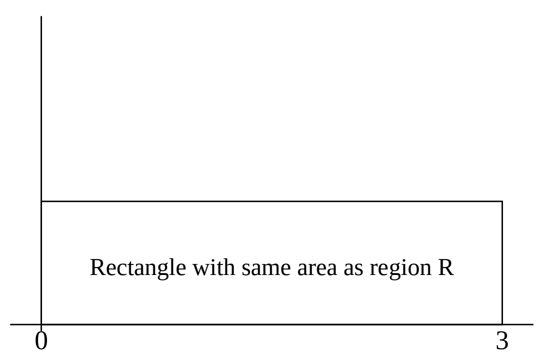
$$x^3\sqrt{1+3x^2}dx = \left(\frac{u-1}{18}\right)\sqrt{u}du = \left(\frac{u^{\frac{3}{2}} - u^{\frac{1}{2}}}{18}\right)du.$$

$u = 0 + 1 = 1$ when $x = 0$, and $u = 1 + 3 = 4$ when $x = 1$. Therefore the definite integral becomes

$$\begin{aligned}\int_0^1 x^3\sqrt{1+3x^2}dx &= \frac{1}{18}\int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \left[\frac{1}{18}\left(\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right)\right]_1^4 \\ &= \frac{1}{18}\left(\frac{64}{5} - \frac{16}{3}\right) - \frac{1}{18}\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{116}{270}\end{aligned}$$

16. Let $f(x) = x^2 + 1$, for values of x between 0 and 3. Determine a number c between 0 and 3 such that the rectangle with base $[0, 3]$ and height $f(c)$ has the same area as the region R enclosed by the graph of f, the x - axis and the two lines $x = 0$ and $x = 3$.





(a) **Solution**: The area of the region R is $\int_0^3 \left(x^2+1\right)dx = \left[\frac{x^3}{3} + x\right]_0^3 = 12 - \frac{4}{3} = \frac{32}{3}$. The width of the rectangle is 3, therefore its height must be $\left(\frac{32}{3}\right) \div 3 = \frac{32}{9}$. Now we must find a number c between 0 and 3 where $f(c) = \frac{32}{9}$. It must satisfy the equation $x^2 + 1 = \frac{32}{9}$.

Solving gives $x = \pm\sqrt{\frac{32}{9}}$. We must choose the positive value because c is between 0 and 3. Therefore $c = \sqrt{\frac{32}{9}}$

17. Use the substitution $x = u^3$ to evaluate $\int_1^8 \left(\frac{1}{x + x^{1/3}}\right) dx$.

(a) **Solution**: Let $x = u^3$. Then $\frac{dx}{du} = 3u^2$, which implies that $dx = 3u^2 du$. It follows that

$$\left(\frac{1}{x + x^{1/3}}\right) dx = \frac{3u^2 du}{u^3 + u} = \frac{3udu}{u^2 + 1}$$

When $x = 1$, the substitution equation becomes $1 = u^3$, which implies that $u = 1$. When $x = 8$, the equation becomes $8 = u^3$, which implies that $u = 2$. Therefore the integral becomes

$$\int_1^8 \left(\frac{1}{x + x^{1/3}}\right) dx = \int_1^2 \frac{3udu}{u^2 + 1} = \frac{3}{2}\int_1^2 \frac{2u}{u^2 + 1} du$$

Since the numerator is the derivative of the denominator, this is a log integral and

$$\frac{3}{2}\int_1^2 \frac{2u}{u^2 + 1} du = \left[\frac{3}{2} \ln\left(u^2 + 1\right)\right]_1^2 = \frac{3}{2}(\ln 5 - \ln 2) = \frac{3}{2} \ln\left(\frac{5}{2}\right)$$

18. Use the substitution $x = u^2$ to evaluate $\int_4^9 \left(\frac{1}{x - x^{1/2}}\right) dx$. (The lower limit is changed to 4 to avoid an undefined value of logarithm at 0.)

(a) **Solution**: Let $x = u^2$. Then Then $\frac{dx}{du} = 2u$, which implies that $dx = 2udu$. It follows that

$$\left(\frac{1}{x - x^{1/2}}\right) dx = \frac{2udu}{u^2 - u} = \frac{2du}{u - 1}$$

When $x = 1$, the substitution equation gives $4 = u^2$, therefore $u = 2$. When $x = 9$, the equation gives $9 = u^2$ which implies that $u = 3$. Therefore the integral becomes

$$\int_4^9 \frac{1}{x - x^{1/2}} dx = \int_2^3 \frac{2du}{u - 1} = 2\int_2^3 \frac{1}{u - 1} du = 2\ln 2 - 2\ln 1 = \ln 4$$

19. Let $f(x) = \frac{1}{\sqrt{1 - x}} = (1 - x)^{-1/2}$. Show that

$$f'(0) = \frac{1}{2},\ f''(0) = \frac{(1)(3)}{2^2},\ f'''(0) = \frac{(1)(3)(5)}{2^3}, \ldots,\ f^{(n)}(0) = \frac{(1)(3)(5)\cdots(2n-1)}{2^n}$$

(a) **Solution**: Take several derivatives and look out for a pattern.

20. Use the results of the above exercise to show that the Maclaurin series for $f(x) = \frac{1}{\sqrt{1 - x}}$ is

$$1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,x^n}{(2^n)(n!)}$$

21. Use the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{1-x}}$ in the above exercise to determine the Maclaurin series for $g(x) = \dfrac{1}{\sqrt{1-x^2}}$, then deduce the Maclaurin series for $h(x) = \arcsin x$.

(a) **Solution**: Replace x by t^2 in the above series to get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,t^{2n}}{(2^n)(n!)}$$

Integrate both sides of the equation, (the variable is t), from 0 to x. The result is

$$\int_0^x \frac{1}{\sqrt{1-t^2}}dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,t^{2n}}{(2^n)(n!)}\right) dt$$

The left hand side is $\int_0^x \frac{1}{\sqrt{1-t^2}}dt = \Big[\arcsin t\Big]_0^x = \arcsin x - 0 = \arcsin x$. The right hand side is

$$\begin{aligned}\int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,t^{2n}}{(2^n)(n!)}\right) dt &= \left[t + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,t^{2n+1}}{(2^n)(n!)(2n+1)}\right]_0^x \\ &= x + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,x^{2n+1}}{(2^n)(n!)(2n+1)}\end{aligned}$$

Therefore

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)\,x^{2n+1}}{(2^n)(n!)(2n+1)}$$