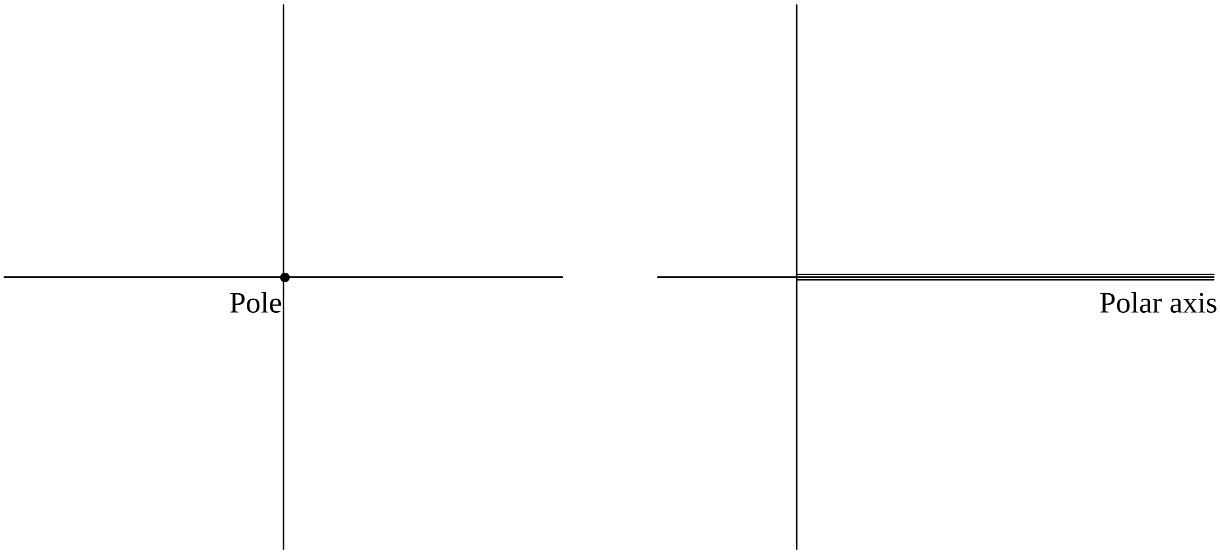
Polar Coordinates

We now introduce another method of labelling points in a plane. We start by fixing a point in the plane. It is called the **pole**. A standard choice for the pole is the origin $(0,0)$ for the Cartezian coordinate system.



We then fix a ray starting from the pole. It is called the **polar axis**. A standard choice for the polar axis is the positive half of the horizontal axis.

To determine the polar coordinates of a given point P in the plane, do the following:

(i) Determine its distance r from the pole.

(ii) Determine the angle θ between the polar axis and the ray OP.

Then the polar coordinates of P are (r,θ).

In the figure below, O is the pole, and OA is the polar axis. The given point P is 2 units from the pole and the ray OP makes an angle of $30°$ with the polar axis. Therefore its polar coordinates are $(2, 30°)$.

There are special polar graph papers that one may easily use to plot points in polar coordinates. The figure below is a simple example. It consists of circles with the same center O and different radii. The common center O is the pole. The ray labelled $0°$ is the polar axis. Rays making angles of $30°$, $60°, \ldots$ are also drawn. Consider the point P marked by *. It is 8 units from the pole, because it is on a circle centered at the pole with radius 8, and the ray OP makes an angle of $30°$ with the polar axis. Therefore P has polar coordinates $(8, 30°)$.

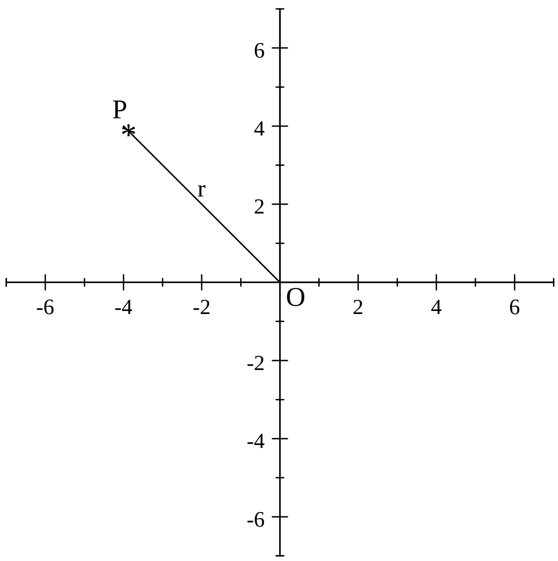
Plot the following points on the above polar graph paper.

a) $(2, 30°)$ b) $(5, 120°)$ c) $(6, 270°)$ d) $(9, 330°)$ e) $(4, 45°)$

f) $(7.5, 110°)$ g) $(8.2, 275°)$ h) $(9.5, 240°)$ i) $(3.6, 220°)$ j) $(1.8, 200°)$

Converting Cartesian coordinates into polar coordinates

For an example, suppose we are required to determine the polar coordinates of the point with Cartesian coordinates $(-4, 4)$. It is plotted in the figure below and it is denoted by P. Its distance r from the origin $(0, 0)$ is $\sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$. The angle θ between the ray OP and the positive horizontal axis should be easy to guess because the ray bisects the right angle between the negative horizontal axis and the positive vertical axis. It is 135° and so the polar coordinates for $(-4, 4)$ are $\left(4\sqrt{2}, 135°\right)$.



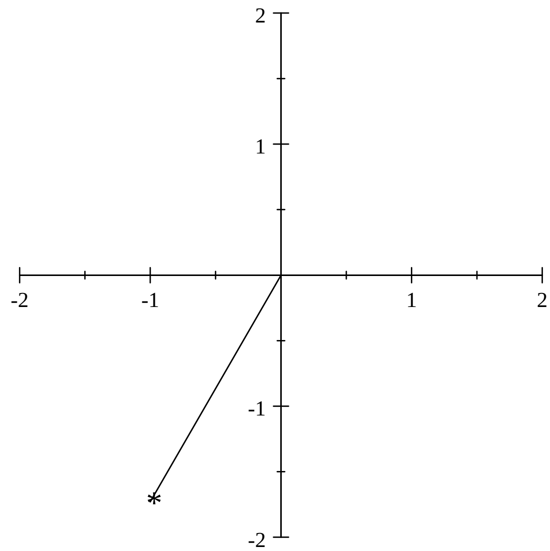
In general, to determine the polar coordinates of a point with Cartesian coordinates (x, y), you have to:

- Determine the distance r from $(0, 0)$ to (x, y). It is given by the formula $r = \sqrt{x^2 + y^2}$.
- Determine the angle between the positive horizontal axis and the ray from the origin $(0, 0)$ to the point (x, y). If the angle is θ then $\tan\theta = \frac{y}{x}$. Determine its value from the fact that it is in the same quadrant as the point (x, y).

Example 1 *To determine the polar coordinates of the point with Cartesian coordinates* $\left(-1, -\sqrt{3}\right)$.

Solution *The point is plotted in the figure below. Its distance r from the origin is given by*

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$



The angle θ *between the positive horizontal axis and the ray from* $(0,0)$ *to* $\left(-1,-\sqrt{3}\right)$ *satisfies the condition* $\tan\theta = \frac{\sqrt{3}}{1}$. *Since it is in the third quadrant,*

$$\theta = (180+60)^\circ = 240^\circ$$

Therefore the polar coordinates for $\left(-1,-\sqrt{3}\right)$ *are* $(2, 240^\circ)$

Exercise 2

1. *Plot the points with the given polar coordinates on the given polar graph*

   a) $(5, 60^\circ)$ *b)* $(7.5, 240^\circ)$ *c)* $(9, 310^\circ)$ *d)* $(10, 200^\circ)$

2. *The polar coordinates for the point with Cartesian coordinates* $(5,5)$ *are*

   (A) $(5, 45^\circ)$ *(B)* $\left(5\sqrt{5}, 45^\circ\right)$ *(C)* $\left(5\sqrt{2}, 45^\circ\right)$ *(D)* $\left(5\sqrt{2}, 60^\circ\right)$

3. *What are the polar coordinates of the point with Cartesian coordinates* $(0,-9)$?

Polar Form Of A Complex Number

The first step in determining the so-called polar form of a given complex number $x+yi$ is to represent it by a point in a plane. So, we draw the usual horizontal and vertical Cartesian coordinates lines as shown below then represented the complex number by the point (x,y).

Example 3 *Consider the complex numbers* $-3+2i$, $-2-4i$, $4.5-2i$ *and* $4+4i$.

- $-3+2i$ *is represented by the point* $*$ *with Cartesian coordinates* $(-3,2)$.
- $-2-4i$ *is represented by the point* $\bullet$ *with Cartesian coordinates* $(-2,-4)$.
- $4.5-2i$ *is represented by the point* $\#$ *with Cartesian coordinates* $(4.5,-2)$.
- $4+4i$ *is represented by the point* $\circ$ *with Cartesian coordinates* $(4,4)$.

The next step is to convert the Cartesian coordinates (x, y) of the point into polar coordinates. Thus we determine $r = \sqrt{a + b^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Example 4 *In the case of $4+4i$ which was represented by the point with Cartesian coordinates $(4,4)$, we get $r = \sqrt{16+16} = 4\sqrt{2}$ and $\theta = \tan^{-1} 1 = \frac{\pi}{4}$ or $45°$, if we choose to measure angles in degrees, and so the polar coordinates of $(4,4)$ are $\left(4\sqrt{2}, \frac{\pi}{4}\right)$ or $\left(4\sqrt{2}, 45°\right)$.*

Now note that the real part of $4+4i$ is $r\cos\frac{\pi}{4} = 4\sqrt{2}\cos\frac{\pi}{4}$ and the complex part is $r\sin\frac{\pi}{4} = 4\sqrt{2}\sin\frac{\pi}{4}$. Therefore

$$4 + 4i = 4\sqrt{2}\cos\tfrac{\pi}{4} + 4\sqrt{2}i\sin\tfrac{\pi}{4} = 4\sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})$$

This is the polar form of $4+4i$.

Going back to an arbitrary complex number $x + yi$, if the polar coordinates for (x, y) are (r, θ), where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore

$$x + yi = r\cos\theta + ri\sin\theta = r\left(\cos\theta + i\sin\theta\right)$$

This is called the **polar form** of the complex number $x + yi$. The distance r is called the absolute value of $x + yi$ and the angle θ is called its argument.

Example 5 *To determine the polar form of the complex number $-3-5i$:*

The number is plotted below.

Its absolute value is $r = \sqrt{3+25} = \sqrt{34}$. Its argument is an angle u in the 3rd quadrant that satisfies the equation $\tan u = \dfrac{-5}{-3} = \dfrac{5}{3}$. Thus the reference angle for u is $\tan^{-1}\left(\dfrac{5}{3}\right) = 59°$, to the nearest degree. It follows that $u = (180+59)^{\circ} = 239°$. The required polar form is

$$-3 - 5i = \sqrt{34}\left(\cos 239° + i\sin 239°\right)$$

Multiplying/Dividing complex numbers in polar form

Consider complex numbers $z = r\left(\cos u + i\sin u\right)$ and $w = l\left(\cos v + i\sin v\right)$ in polar form.

- Their product zw is

$$\begin{aligned} zw &= r\left(\cos u + i\sin u\right) l\left(\cos v + i\sin v\right) = rl\left(\cos u + i\sin u\right)\left(\cos v + i\sin v\right) \\ &= rl\left[\left(\cos u\cos v - \sin u\sin v\right) + i\left(\sin u\cos v + \cos u\sin v\right)\right] = rl\left[\cos(u+v) + i\sin(u+v)\right] \end{aligned}$$

Note that the absolute value of the product is rl which is the product of the absolute values of z and w. The argument of zw IS NOT the product of the two arguments; it is $u + v$, which is the sum of the two argument.

Conclusion: To multiply two complex numbers in polar form, simply multiply their absolute values, then add their arguments.

Example: The product of $z = 4\left(\cos 42° + i\sin 42°\right)$ and $w = 3.1\left(\cos 165° + i\sin 165°\right)$ is the complex number

$$zw = (4)(3.1)\left[\cos\left(42+165\right)^{\circ} + i\sin\left(42+165\right)^{\circ}\right] = 12.4\left(\cos 207° + i\sin 207°\right)$$

- The quotient $\dfrac{z}{w}$ is

$$\begin{aligned}
\frac{z}{w} &= \frac{r(\cos u + i\sin u)}{l(\cos v + i\sin v)} = \left(\frac{r}{l}\right)\frac{(\cos u + i\sin u)}{(\cos v + i\sin v)} = \left(\frac{r}{l}\right)\frac{(\cos u + i\sin u)(\cos v - i\sin v)}{(\cos v + i\sin v)(\cos v - i\sin v)} \\
&= \left(\frac{r}{l}\right)\frac{(\cos u + i\sin u)(\cos v - i\sin v)}{(\cos v + i\sin v)(\cos v - i\sin v)} = \left(\frac{r}{l}\right)\frac{[(\cos u\cos v + \sin u\sin v) + i(\sin u\cos v - \cos u\sin v)]}{(\cos^2 v + \sin^2 v)} \\
&= \frac{r}{l}[\cos(u-v) + i\sin(u-v)]
\end{aligned}$$

Thus the absolute value of $\dfrac{z}{w}$ of is $\dfrac{r}{l}$ and its argument is $u - v$.

Conclusion: To divide a complex number z by a complex number w, both in polar form, divide the absolute value of z by the absolute value of w, then subtract the argument of w from the argument of z.

Example: Let $z = 18(\cos 123° + i\sin 123°)$ and $w = 2.4(\cos 65° + i\sin 65°)$. Then

$$\frac{z}{w} = \frac{18}{2.4}\left[\cos(123-65)^\circ + i\sin(123-65)^\circ\right] = \frac{15}{2}(\cos 58° + i\sin 58°)$$

Example: Let $z = 2.1(\cos 60° + i\sin 60°)$ and $w = 2.4(\cos 105° + i\sin 105°)$. Then

$$\begin{aligned}
\frac{z}{w} &= \frac{2.1}{2.4}\left[\cos(60-105)^\circ + i\sin(60-105)^\circ\right] = \frac{7}{8}(\cos(-45°) + i\sin(-45°)) \\
&= \frac{7}{8}(\cos 45° - i\sin 45°) = \frac{7}{8}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)
\end{aligned}$$

Example 6 *To find the square roots of $z = -4i$. In other words, we must find a complex number whose square is $-4i$*

The first step is to write $-4i$ in polar form: Its absolute value is 4 and its argument is $270°$. Therefore $-4i = 4(\cos 270° + i\sin 270°)$. Let its square root be $w = r(\cos u + i\sin u)$. In other words, let w be a number that satisfies $w^2 = -4i$. We note that

$$w^2 = ww = r^2(\cos 2u + i\sin 2u)$$

Therefore

$$r^2(\cos 2u + i\sin 2u) = 4(\cos 270° + i\sin 270°)$$

It follows that $r^2 = 4$ and $2u = 270°$. Therefore $r = 2$ and $u = 135°$. Conclusion: A square root of $-4i$ is $2(\cos 135° + i\sin 135°) = 2\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\sqrt{2} + \sqrt{2}i$. (Square it and see what you get.)

To get a second square root, write $-4i$ as $4\left(\cos(270+360)^\circ + i\sin(270+360)^\circ\right) = 4(\cos 630° + i\sin 630°)$. Now

$$r^2(\cos 2u + i\sin 2u) = 4(\cos 630° + i\sin 630°)$$

Following the above foot-steps, we get $r^2 = 4$ and $2u = 630°$. This gives $r = 2$ and $u = 315°$. Therefore another square root of $-4i$ is $2(\cos 315° + i\sin 315°) = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2} - \sqrt{2}i$.

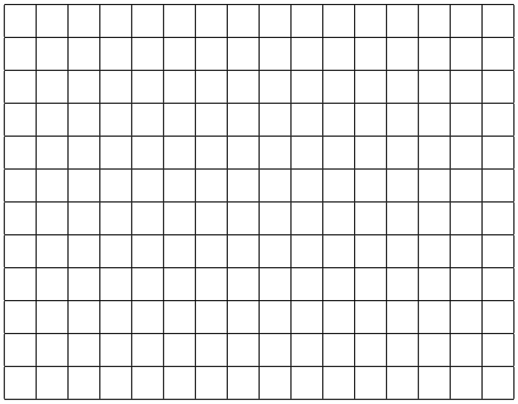
Exercise 7

1. *Plot the following points on the polar graph paper below.*

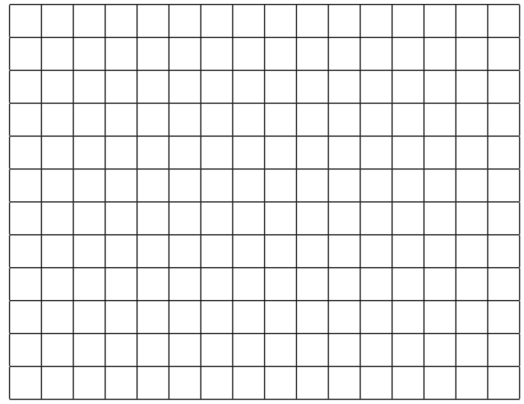
   a) $(6, 75°)$ *b)* $(9.5, 270°)$ *c)* $(9, 320°)$ *d)* $(5, 205°)$

2. *You are given the complex numbers $z_1 = 4 - 4i$, $z_2 = -\sqrt{3} - i$, $z_3 = 1 - \sqrt{3}i$, $z_4 = 5 + 5i$.*

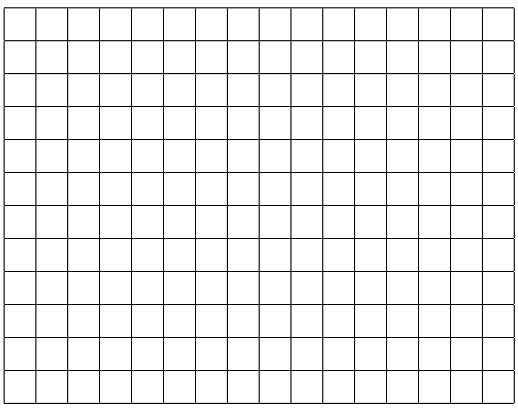
(a) Draw axes and plot each number.



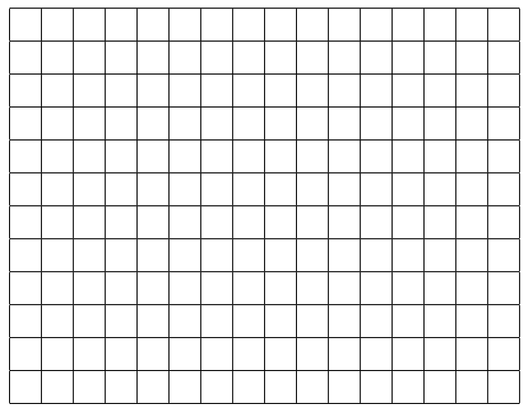
$4 - 4i$



$-\sqrt{3} - i$



$1 - \sqrt{3}i$



$5 + 5i$

(b) Determine the polar form of the numbers z_1, z_2, z_3 *and* z_4 *above.*

(i)

Absolute value of z_2 *is*

Argument of z_2 *is*

Polar form of z_2 *is*

(ii)

Absolute value of z_3 *is*

Argument of z_3 *is*

Polar form of z_3 *is*

(iii)

Absolute value of z_4 *is*

Argument of z_4 *is*

Polar form of z_4 *is*

(c) *Determine* $z_1 z_2$ *and leave your answer in polar form*

$z_1 z_2 =$

(d) *Determine* $\frac{z_2}{z_3}$ *and leave your answer in polar form.*

$\frac{z_2}{z_3} =$

(e) *Determine* $z_1 z_4$ *and leave your answer in polar form*

$z_1 z_4 =$

3. *What is the polar form of the complex number* $z = 2 - \left(2\sqrt{3}\right) i$*?*

   (A) $2\left(\cos 120^\circ + i \sin 120^\circ\right)$ (B) $4\left(\cos 60^\circ + i \sin 60^\circ\right)$ (C) $2\left(\cos 300^\circ + i \sin 300^\circ\right)$

   (D) $4\left(\cos 300^\circ + i \sin 300^\circ\right)$

4. *You are given the complex number* $z = 2 + \left(2\sqrt{3}\right) i$.

   (a) *Determine the polar form of* z.

   (b) *Find the two square roots of* z. *You may give your answers in polar form.*