Areas of some geometric figures

The area of a given geometric figure is the number of unit squares, (i.e. squares with length and width equal to 1 unit), that may be fitted into the region.

The simplest is a rectangle like the one in figure (i) below. It has length 6 cm and width 5 cm. Its area, in square centimeters, is the number of squares with length and width equal to 1 cm each, that can be fitted into the rectangle.

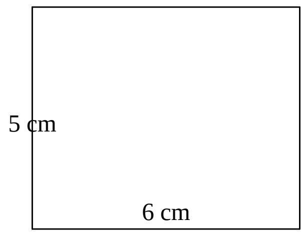


Figure (i)

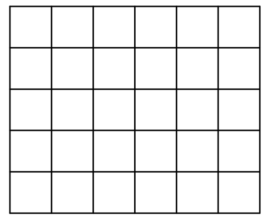
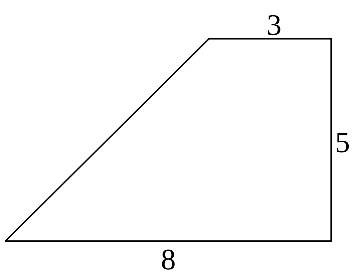


Figure (ii)

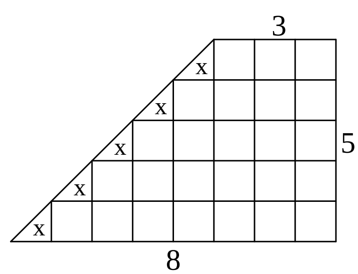
As shown in Figure (ii), they are 30, therefore its area is 30 square centimeters, abbreviated to 30 sq. cm.

It should be easy to see that the area of a rectangle with length l units and width w units is lw sq. units.

Of course areas do not have to be whole numbers. The area of the trapezium below is $27\frac{1}{2}$



A trapezium



The trapezium sliced into squares

We sliced it into unit squares as shown in the figure to the right. There are 25 full squares plus 5 half squares for a total of $27\frac{1}{2}$ square centimeters.

The next simple figure to consider is a parallelogram like the one shown in Figure (iii) below. It has vertices at $(0, 0)$, $(5, 0)$, $(8, 4)$ and $(3, 4)$.

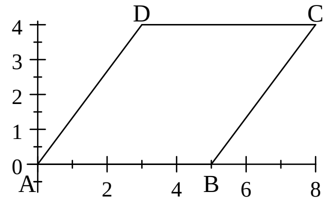


Figure (iii)

To figure out the number of unit squares that fit into it, cut out triangle AEC, shown in figure (iv) and paste it as shown in figure (v). The result is a rectangle with corners at $(3,0)$, $(8,0)$, $(8,4)$ and $(3,4)$ shown in Figure (vi). A total of $5 \times 4 = 20$ unit squares fit into the rectangle, therefore the parallelogram has area 20 square units.

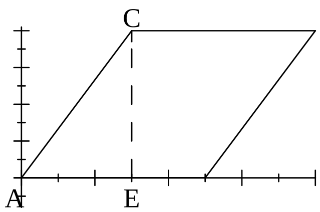


Figure (iv)

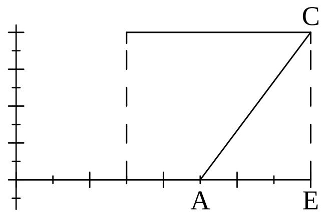


Figure (v)

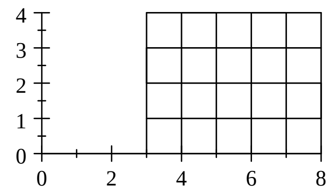


Figure (vi)

We now introduce some terms used in computing areas of parallelograms: Referring to the parallelogram ABCD in Figure (iii),

- The side AB is called a base for the parallelogram. In this case it has length 5 units.
- The length of the line segment EC is called the height of the parallelogram. Thus a height of a parallelogram is the distance between two parallel sides of the parallelogram. The parallelogram in this case has geight 4.

Note that the area of this parallelogram is $5 \times 4 = 20$ square unit which happens to be the product of its length and its height. In general,

$$\text{Area of a parallelogram} = (\text{Length of parallelogram}) \times (\text{Its Height})$$

The third figure we consider is a triangle. Figure (a) shows a triangle ABC. We pick one of its three sides and call it the base of the triangle. For convenience, we have picked AB. We then drop a perpendicular from the vertex opposite the base, which is C, to the base itself. This is the line segment CD in Figure (b). The length of CD is called the height of the triangle. The triangle may be converted into a parallelogram ABEC, shown in Figure (c), as follows: Imagine that BC is a mirror. Now draw a reflection of triangle ABC in the mirror. The result is the parallelogram.

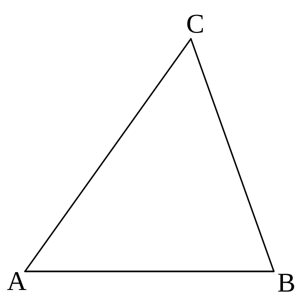


Figure (a)

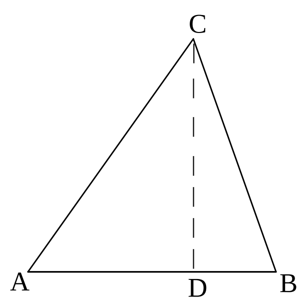
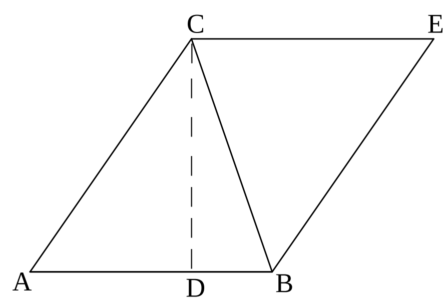


Figure (b)



The base of the parallelogram is AB, and its height is the length of CD, therefore its area is equal to

$$(\text{Length of base AB}) \times (\text{Length of CD})$$

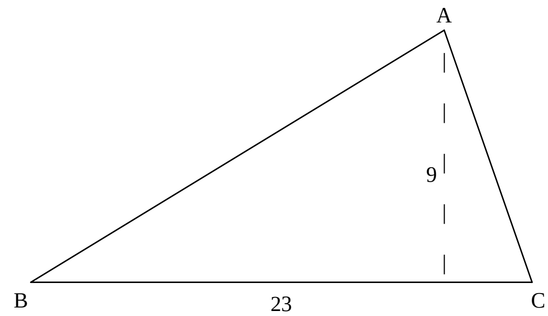
Note that the area of the parallelogram ABEC is twice the area of the triangle ABC. Therefore

$$\text{Area of triangle ABC} = \frac{1}{2} \times (\text{Length of the Base of ABC}) \times (\text{Height of ABC})$$

It turns out that this result holds for any given triangle. In other words:

$$\text{The Area of a Triangle} = \frac{1}{2} \times (\text{Length of its Base}) \times (\text{Its Height})$$

Example 1 *The area of triangle ABC, in the figure below, with a base of length 21 cm and height 9 cm is* $\frac{1}{2}(23 \times 9) = 103.5$ *sq. cm.*



The height may not be given. In such a case, use the given information to calculate it.

Example 2 *In triangle ABC below, $A = 48°$, $b = 18$ cm and $c = 21$ cm. Take AB as the base of the triangle. It has length 21 cm.*

(i) *(ii)*

In figure (ii), we have added the height CD of the triangle. Since

$$\frac{\text{Length of } DC}{\text{Length of } AC} = \sin 48°$$

it follows that the height of the triangle is $(\text{Length of } AC) \times (\sin 48°) = (18)(0.7431)$. *Therefore*

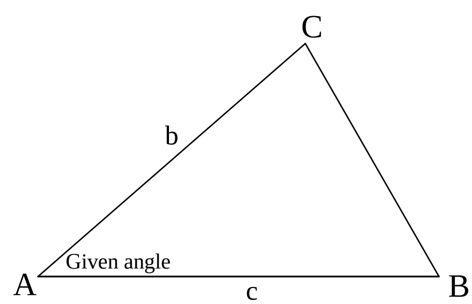
$$\text{Area of Triangle } = \tfrac{1}{2} \times (21) \times (18)(0.7431) = 140.45 \text{ sq. cm.}$$

We rounded off the answer to 2 decimal places.

Area of a SAS triangle

In general, to calculate the area of a given SAS triangle ABC, follow the steps described above. Say you are given the angle A and the two sides b and c as shown below. Take AB as the base. The length of the base is c and the height of the triangle is $b \sin A$. Therefore the area of the triangle is

$$\boxed{\tfrac{1}{2} bc \sin A}$$



Example 3 *To calculate the area of triangle ABC with $a = 12$ ft., $C = 67°$ and $b = 9$ ft. and round off the answer to 2 dec. pl.*

Solution *We are given two sides with lengths 12 ft. and 9 ft. We are also given the angle C between these two sides. It is $C = 67°$. Therefore the area of the triangle is*

$$\text{Area} = \tfrac{1}{2} \times 12 \times 9 \times \sin 67° = 49.71$$

Area of a SAA triangle

If you are given two angles and a side, (i.e. you are given a SAA triangle), calculate one of the unknown sided. You will then have two sides and an angle between them and you may calculate the area of the triangle as described above.

Example 4 *To calculate the area of triangle ABC with $A = 43°$, $B = 72°$ and $b = 18$ cm.*

Solution *If we calculate c then we will have two sides and an angle between them. (The two sides are c and b; the angle between them is A.) Since $C = (180 - 72 - 43)^{\circ} = 65°$, c is given by*

$$\frac{c}{\sin 65°} = \frac{18}{\sin\ 72°}.$$

It follows that $c = \dfrac{18 \sin 65°}{\sin\ 72°}$. *The area of the triangle is $\frac{1}{2} bc \sin A$ which translates into*

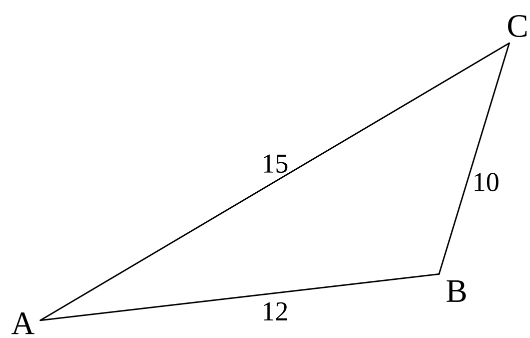
$$\tfrac{1}{2} \times 18 \left(\frac{18 \sin 65°}{\sin\ 72°} \right) \sin 43° = 154.4 \textit{ sq. cm. to 1 decimal place.}$$

Area of a SSS triangle

If you are given the lengths of the three sides of a triangle ABC and no angle, you may calculate its area using Heron's formula. Say the sides have lengths a, b, and c. Start by determining the number $s = \frac{1}{2}(a + b + c)$. Then according to Heron's theorem, the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

- **Example 5** *To calculate the area of the triangle ABC with $a = 10$ cm, $b = 15$cm and $c = 12$ cm.*

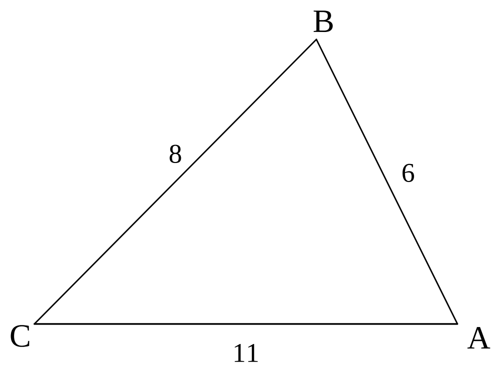


Solution: *In this triangle, $s = \frac{1}{2}(12 + 10 + 15) = 18.5$. Therefore $s - a = 8.5$, $s - b = 3.5$ and $s - c = 6.5$. By Heron's formula, the area of the triangle is*

$$\sqrt{(18.5)(8.5)(3.5)(6.5)} = 59.8 \textit{ square centimeters, (to 1 decimal place)}$$

Exercise 6 *Calculate the area of each triangle*

(i) *(ii)*



(iii) *(iv)*