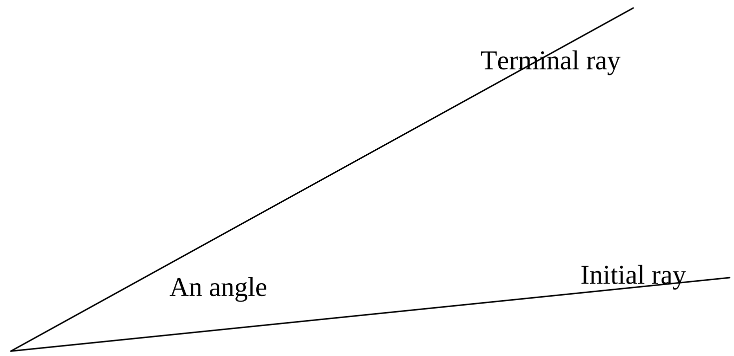
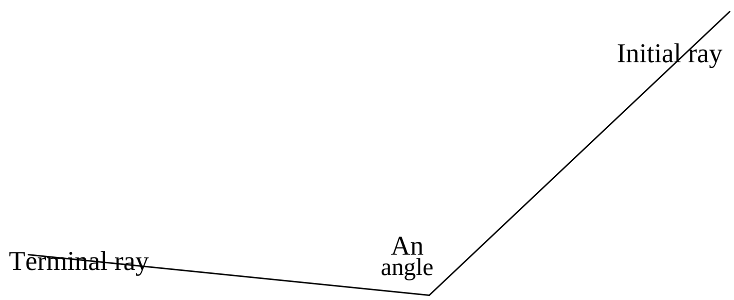
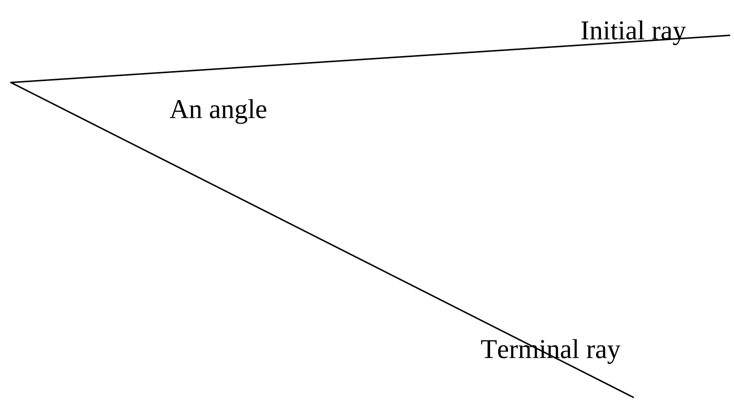
Angles

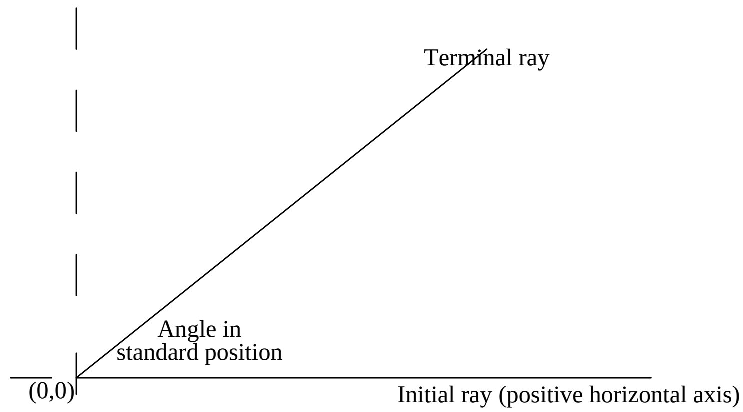
An angle is formed when two rays intersect, (see the figures below). One of the rays is called the initial ray and the other one the terminal ray.



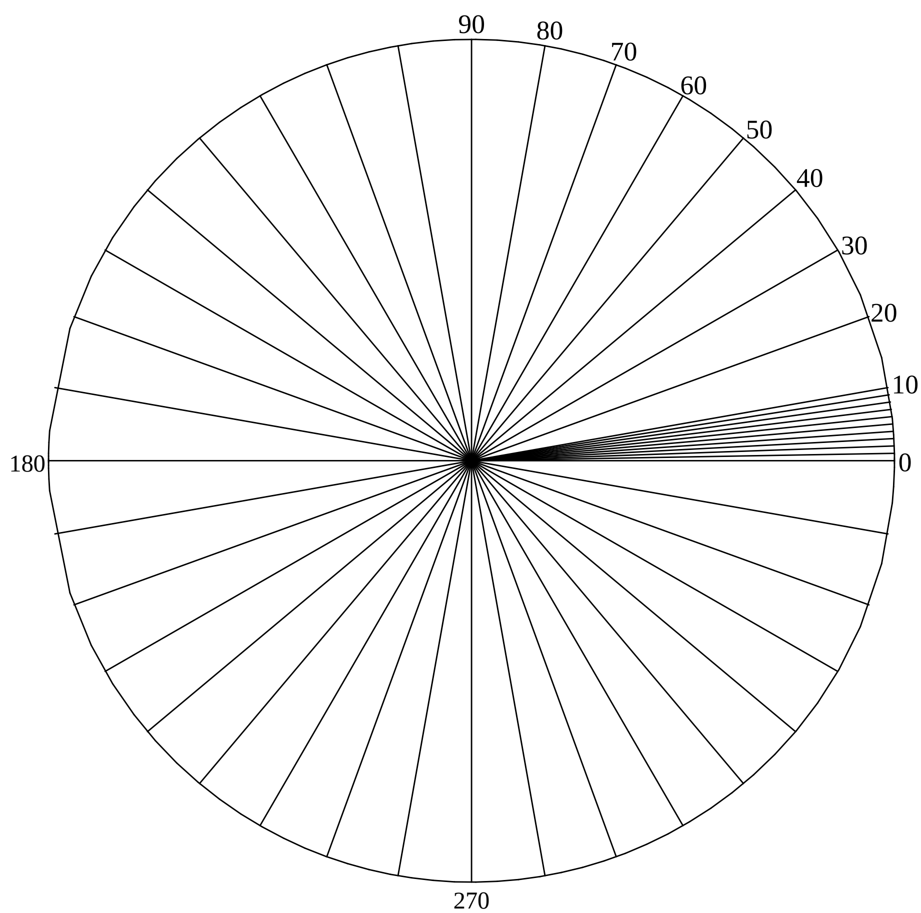




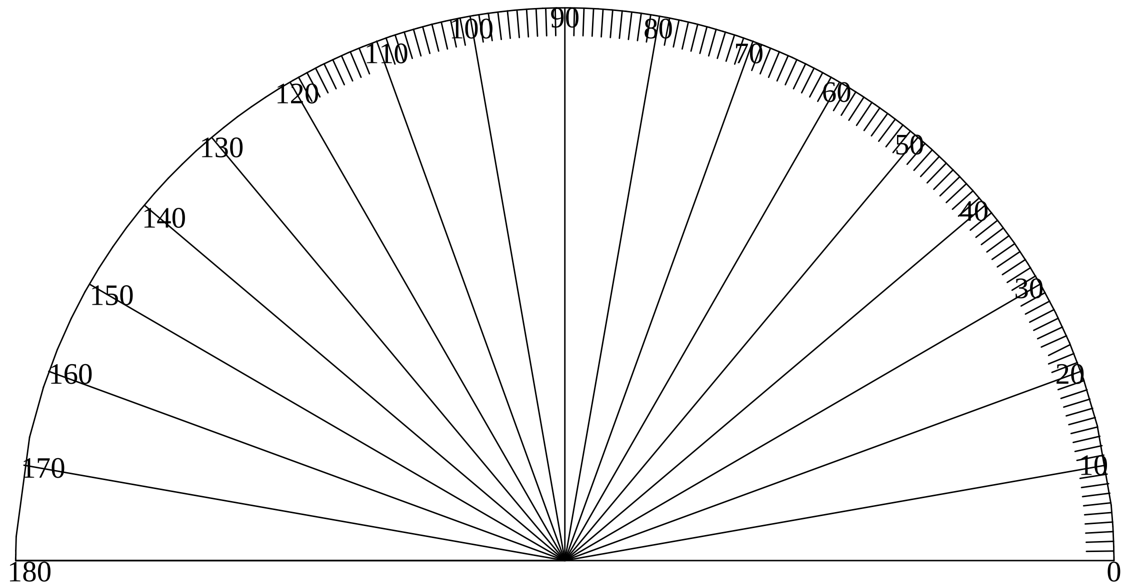
A given angle is in standard position if the initial and terminal ray originate from the origin $(0,0)$ of the Cartesian coordinate system and the initial ray is the positive horizontal axis.



A common measure of the size of an angle is a **degree**. To visualize it, imagine drawing 360 equally spaced rays on a circle. The figure below shows ten of them, between the 0 and 10 marks. One degree, (written as 1°), is the angle between any two such rays that are next to each other. It is a pretty small angle.



A picture of a simple instrument for measuring is shown below. It is called a protractor. It is just a transparent semicircle with 180 equally spaced rays.



A protractor

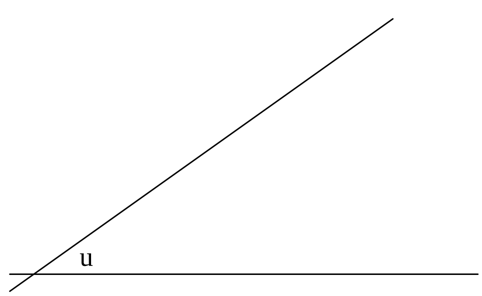
You may use it to measure angles between 0 and 180 degrees directly. For angles between 180 and 360 degrees, first measure off 180 degrees then determine the size of the remaining part of the angle, (which should be less than 180 degrees), and add it to the 180 degrees. The back page has protractors you may cut out for use in estimating angles.

- An angle measuring 90°, shown below, is called a right angle.

  A right angle

  One line is said to be perpendicular to another line if the angle between them is 90°.

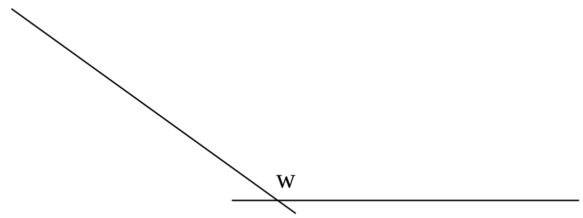
- An angle measuring between 0° and 90° is called an acute angle.



An acute angle u in standard position

- The figure below shows a 180° angle.

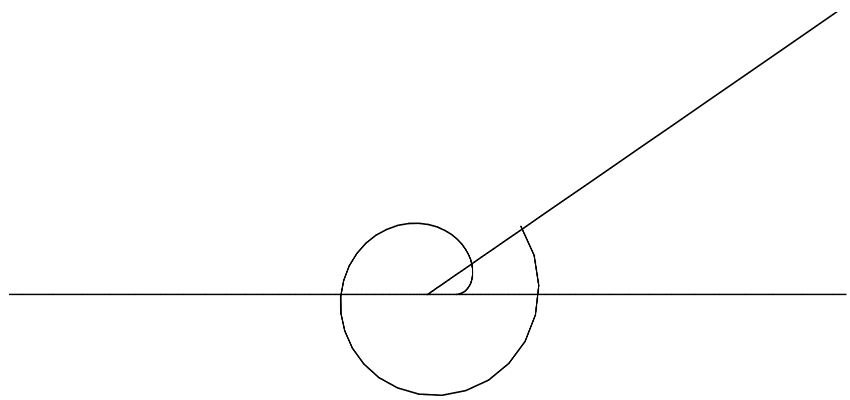
- An obtuse angle is an angle measuring between 90° and 180°.



An obtuse angle w in standard position

- A full revolution is equal to 360°.

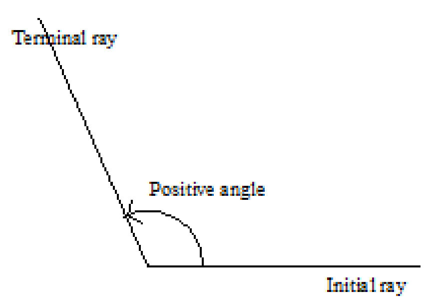
- Angles can be bigger than 360° because it is possible to turn more than one revolution.



An angle bigger than 360°

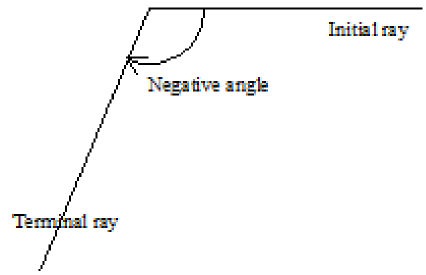
Positive and Negative Angles

To draw an angle, you imagine starting with two coincident rays then rotate one of them to the required position. If you rotate it counter-clockwise, the resulting angle is positive.



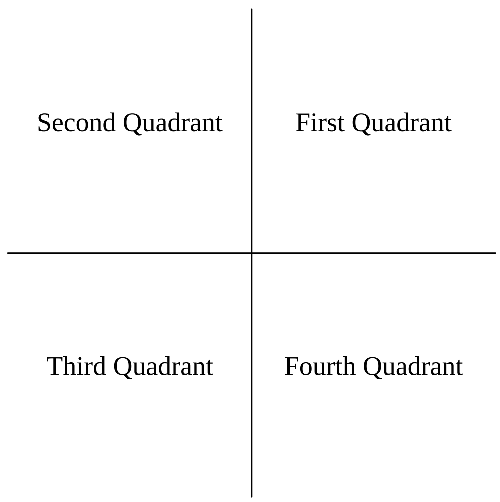
A positive angle in standard position

The arrow indicates the direction of rotation, therefore we need not point out the initial and finaI rays. If you rotate it clockwise, the resulting angle is negative.

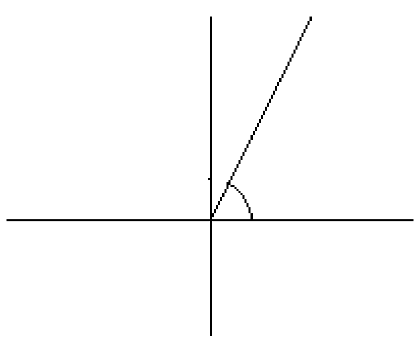


A negative angle in standard position

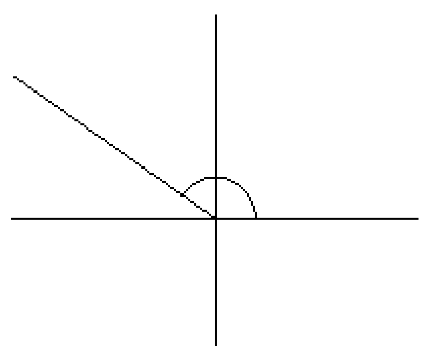
For many problems, we will need to draw angles in the Cartezian coordinate plane. The vertical and horizontal axes divide it into four "quadrants" shown in the figure below.



An angle in standard position is in the first quadrant if its terminal ray is in first quadrant. It is in the second quadrant if its terminal ray is in the second quadrant. (See the figures below.) For example, all the angles between 0° and 90° are in the first quadrant and all angles between 90° and 180° are in the second quadrant.

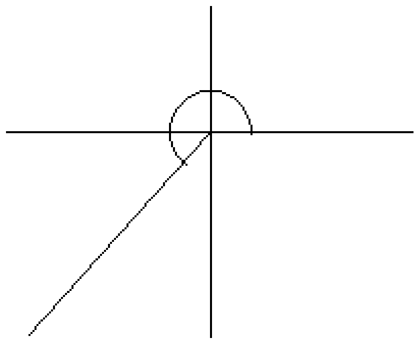


An angle in the first quadrant

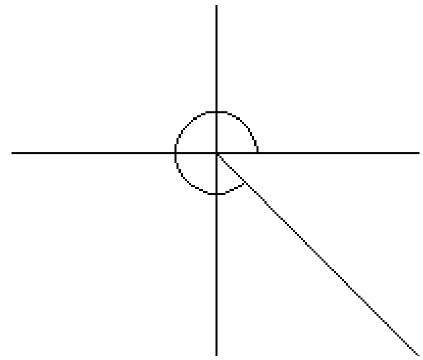


An angle in the second quadrant

The angles in the third quadrant have their terminal ray in the third quadrant and those in the fourth quadrant have their terminal ray in the fourth quadrant, see the figures below. In particular, angles between 180° and 270° are in the third quadrant and angles between 270° and 360° are in the fourth quadrant.



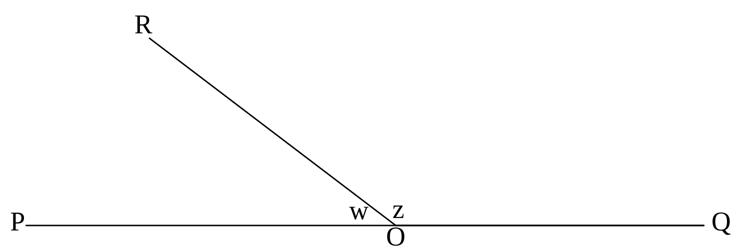
An angle in the third quadrant



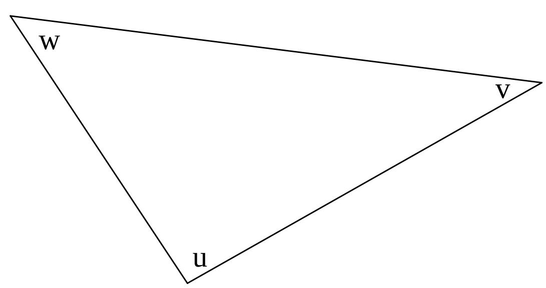
An angle in the fourth quadrant

You must have encountered angles in geometric figures like triangles, rectangles and more general polygons. In these circumstances, it is the absolute values of the angles that are used, thus all the angles are positive. The following facts are most probably familiar:

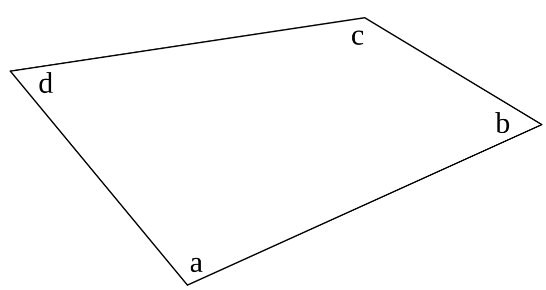
- In the figure below, PQ is a straight line. The sum of the two angles w and z is 180°.



- The sum of the angles of a triangles is 180°. Thus, in the triangle below, $u + v + w = 180°$



- The sum of the interior angles of a 4 sided polygon is 360°. Therefore, in the four sided figure below, $a + b + c + d = 360°$



Example 1 *Consider the figure below.*

Since $65° + v = 180°$, *it follows that* $v = 180° - 65° = 115°$.

In the case of w,

$$w + 148° = 180°$$

It follows that $w = 180° - 148° = 32°$.

To find u, *we use the fact that the angles of a triangles add up to* $180°$. *Thus*

$$65° + w + u = 65° + 32° + u = 180°$$

Therefore $u = 180° - 97° = 83°$.

Lastly, z *must be* $97°$ *because* $z + u = z + 83° = 180°$.

Example 2 *In the figure below,* $a = 40°$ *because* $a + 140° = 180°$.

Since the interior angles of a 4-sided figure add up to 360°, it follows that

$$360° = a + 159° + 59° + b = 40° + 159° + 59° + b$$

Therefore $b = 102°$

Since $b + c = 180°$ *and* $b = 102°$, *it follows that* $c = 78°$.

Exercise 3

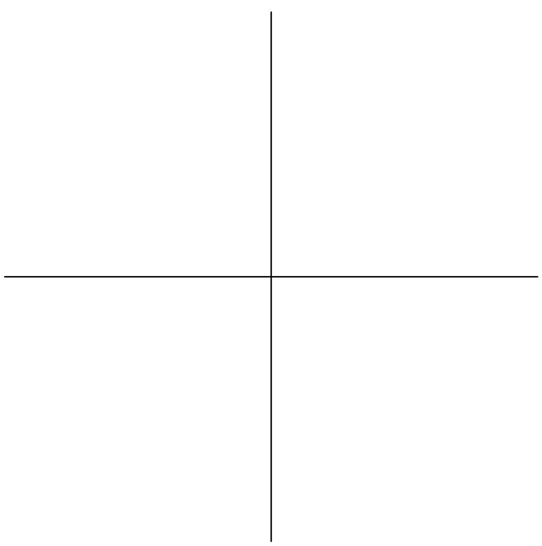
1. *Determine the unknown angles in each figure below.*

Figure (i) *Figure (ii)*

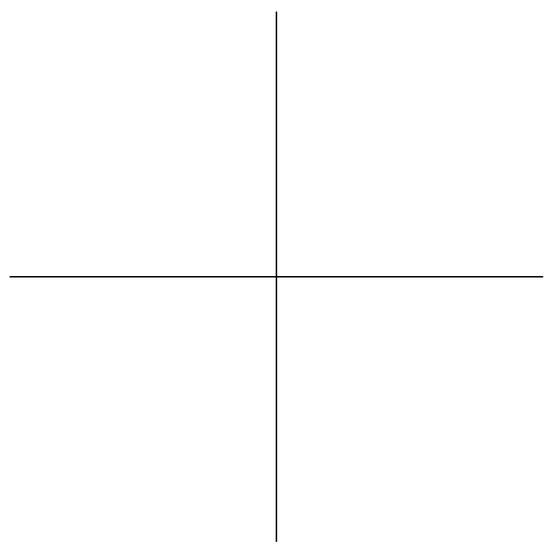
2. *What is the sum of the angles a, b, c and d in figure (ii) above?*

Practice Problems Set 1, v1

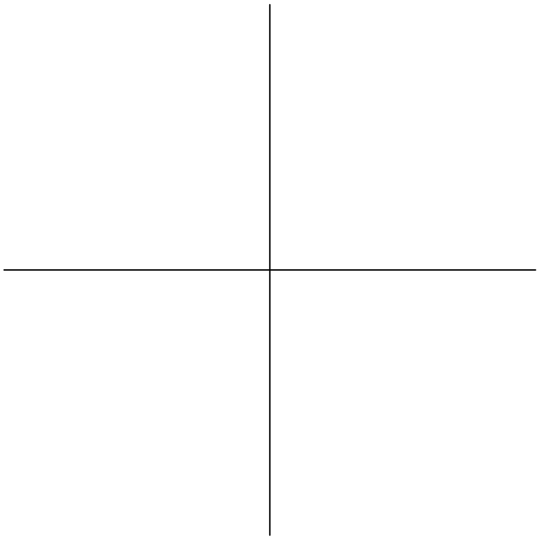
1. Draw the given angle in standard position. It does not have to be accurate, but a reasonable estimate is necessary. In each case use an arrow to indicate the direction of rotation.



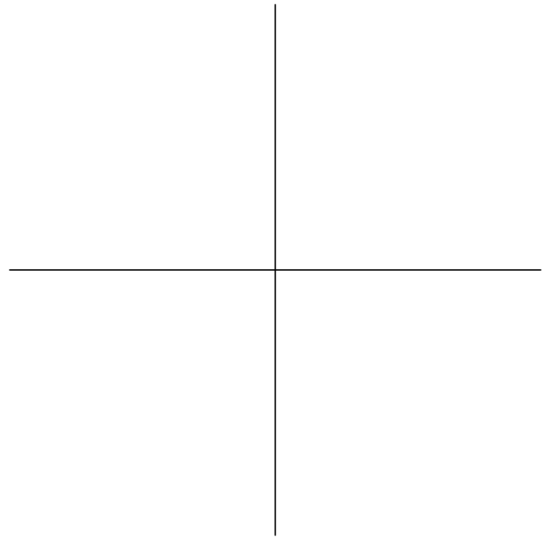
An angle of $45°$



An angle of $-120°$



An angle of 425°



An angle of -385°

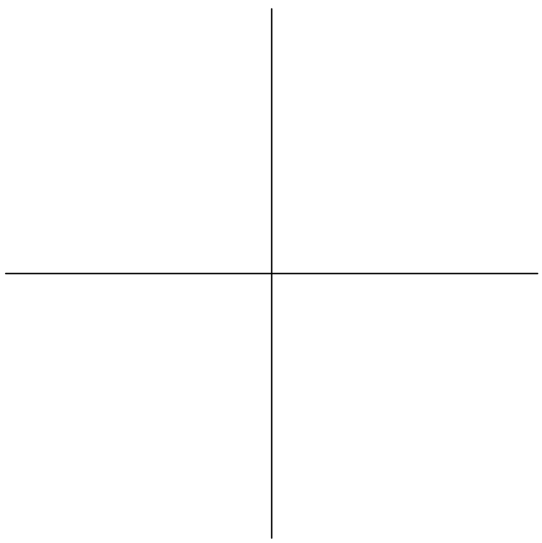
2. Determine the unknown angles in each figure. To get full credit, you must decsribe in your own words, how you arrive at your answers.

   (a)

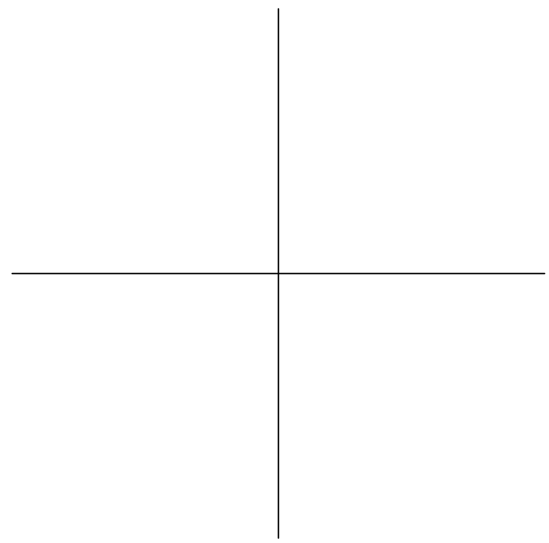
(b)

Practice Problems Set 1, v2

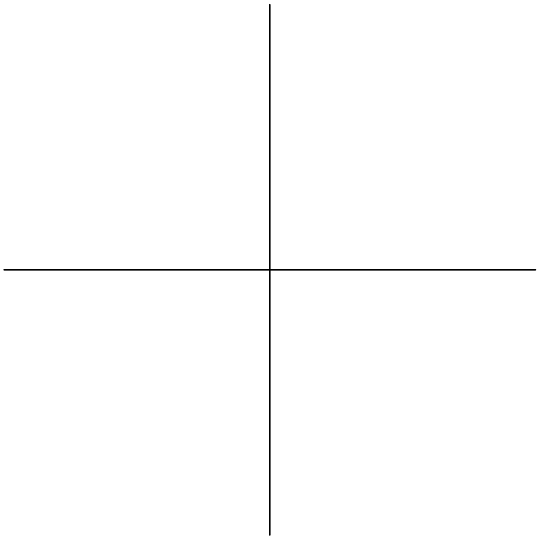
1. Draw the given angle in standard position. It does not have to be accurate, but a reasonable estimate is necessary. In each case use an arrow to indicate the direction of rotation.



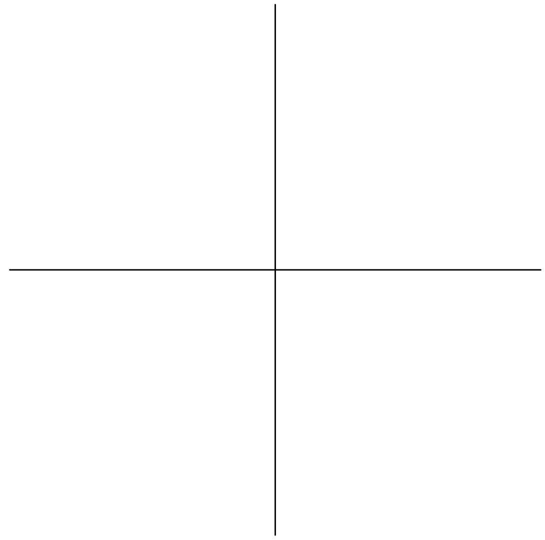
An angle of $60°$



An angle of $-135°$



An angle of $405°$



An angle of $-420°$

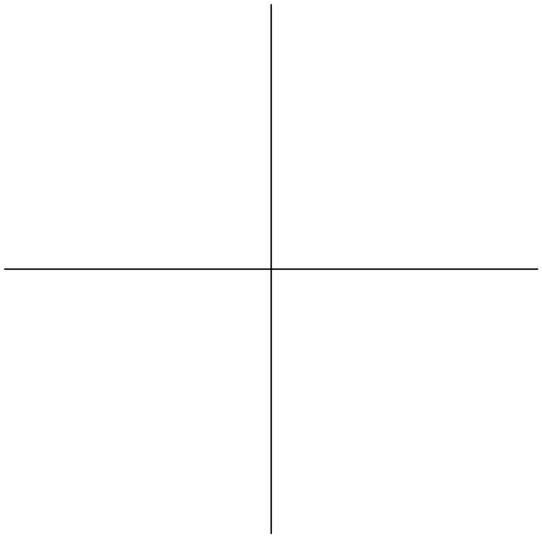
2. Determine the unknown angles in each figure. To get full credit, you must decsribe in your own words, how you arrive at your answers.

   (a)

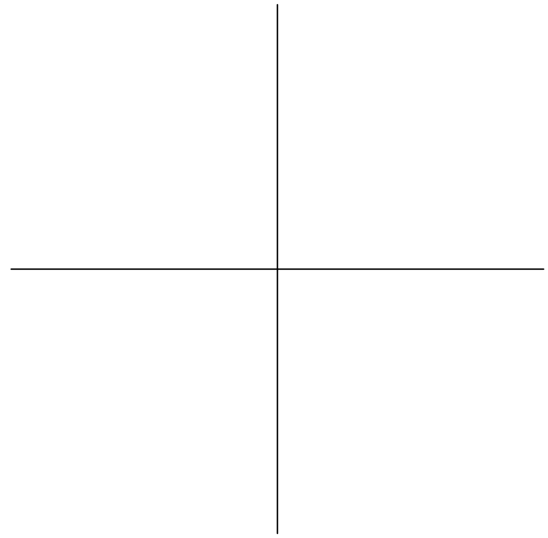
   (b)

Practice Problems Set 1, v3

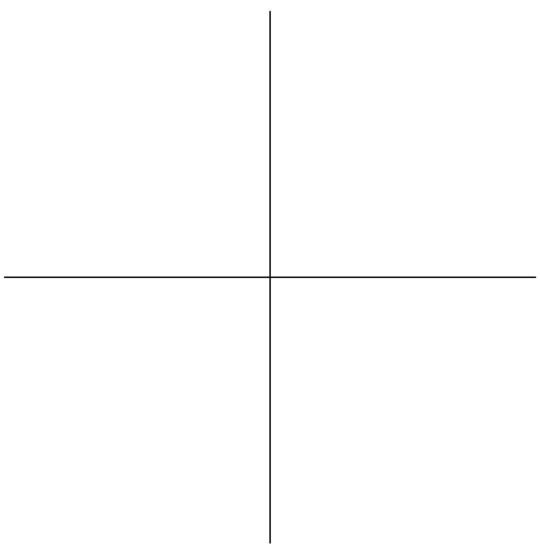
1. Draw the given angle in standard position. It does not have to be accurate, but a reasonable estimate is necessary. In each case use an arrow to indicate the direction of rotation.



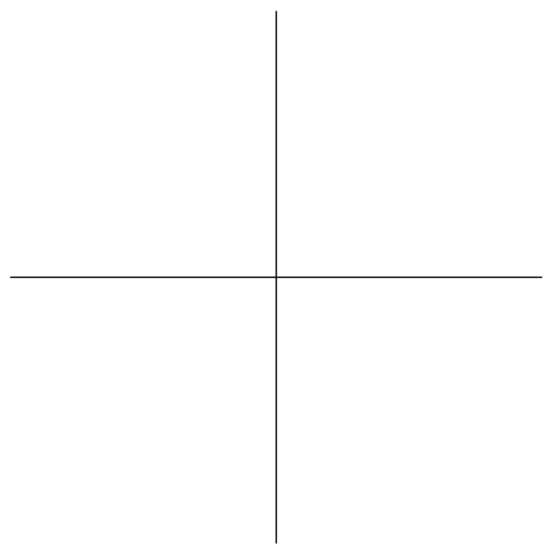
An angle of $45°$



An angle of $-120°$



An angle of $425°$



An angle of $-385°$

2. Determine the unknown angles in each figure. To get full credit, you must decsribe in your own words, how you arrive at your answers.

   (a)

   (b)

Smaller units of angles

There are applications where smaller units of angle measures are required. Two such units are the minute and the second, (not to be confused with the **minutes** and **seconds** we use to measure time). A minute is obtained by dividing a degree into 60 equal parts. Each one of the resulting unit is called a minute. One minute is written as $1'$. Thus

$$1' = \frac{1^\circ}{60}. \qquad \text{Alternatively,} \qquad 1^\circ = 60'$$

When writing an angle measured in degrees and minutes, the degrees are written first followed by the minutes. For example, an angle of 59 degrees and 33 minutes is written as

$$59^\circ 33'$$

We may convert $59^\circ 33'$ into decimal degrees. We simply divide the minuted by 60, thus

$$59^\circ 33' = \left(59 + \frac{33}{60}\right) \text{ degrees which we may write as } 59.55^\circ$$

A **second** is obtained by dividing a minute into 60 equal parts. Each one of the resulting unit is called a second and one second is denoted by $1''$. Thus

$$1'' = \frac{1'}{60}. \qquad \text{Alternatively,} \qquad 1' = 60''$$

When writing an angle measured in degrees, minutes and seconds, the degrees are written first followed by the minutes and then the seconds. For example, an angle of 120 degrees, 24 minutes and 54 seconds is written as

$$120°24'54''$$

Since 1 degree is equal to 60 minutes and 1 minute equals 60 seconds, it follows that 1 degree equals 3600 seconds. I.e.

$$1° = 3600'' \qquad \text{OR} \qquad 1'' = \frac{1°}{3600}$$

We use this, plus the fact that $1' = \dfrac{1°}{60}$, to convert an angle in degrees, minutes and seconds into an angle in decimal degrees.

Example 4 *To convert* $141°30'18''$ *into decimal degrees:*

Solution *There is nothing to do about* $141°$ *because it is already in degrees. It is the* $30'$ *and the* $18''$ *that have to be converted. Since* $60'$ *equal* $1°$*,*

$$30' = \frac{30}{60} = 0.5 \quad \textit{degrees}$$

Since $3600''$ *equal* $1°$*,*

$$18'' = \frac{18}{3600} = 0.005 \ \textit{degrees}$$

Therefore

$$141°30'18'' = 141 + 0.5 + 0.005 = 141.505 \quad \textit{degrees}$$

If you have a calculator you would simply input

$$141 + 30 \div 60 + 18 \div 3600$$

and press an appropriate button, (depending on your calculator), to get the result 141.505.

Example 5 *To convert* $120°24'54''$ *into decimal degrees:*

Solution

$$120°24'54'' = 120 + \frac{24}{60} + \frac{54}{3600} \ \textit{degrees.} \quad \textit{I.e.} \quad 120°24'54'' = 120.415°$$

Example 6 *To convert* $40°35'22''$ *into decimal degrees and round off the answer to 4 decimal places:*

Solution

$$40°35'22'' = \left(40 + \frac{35}{60} + \frac{22}{3600}\right)^{\circ} = 40.589444°$$

When we round this to 4 decimal places, the result is $40°35'22'' = 40.5894°$ *to 4 dec. pl.*

In other instances, we may want to convert an angle in decimal degrees into an angle in degrees, minutes and seconds. For example, say we want to convert 69.7354° into degrees minutes and seconds. We set aside the 69 full degrees and convert the decimal part 0.7354° into minutes. We simply multiply it by 60 because 1° equals 60 minutes. The result is

$$0.7354° = (0.7354 \times 60)' = 44.124'$$

We then set aside the 44 full minutes and convert the decimal part $0.124'$ into seconds by simply multiplying it by 60, (not 3600 because 1 minute equals 60 seconds). The result is

$$0.124' = (0.124 \times 60) = 7.44 \ \text{ seconds}$$

Therefore,

$$69.7354° = 69°44'7.44''$$

It is common to round off to the nearest second. If we are required to do that, we would write

$$69.7354° = 69°44'7'' \ \text{ to the nearest second.}$$

Example 7 *To convert* 248.7873° *into degrees, minutes and seconds, to the nearest second.*

Solution *We convert* 0.7873° *into minutes. The result is*

$$0.7873 \times 60 = 47.238 \quad \textit{minutes}$$

We then convert 0.238′ *into seconds. The result is*

$$0.238 \times 60 = 14.28 \quad \textit{seconds}$$

We round this off to 14 *seconds Therefore*

$$248.7873^\circ = 248^\circ 47' 14''$$

Exercise 8

1. $320^\circ 36' 45''$ *in decimal degrees is equal to:*

   A) 320.6125° *B) 320.6135°* *C) 320.6225°* *D) 320.6145°*

2. *Convert* $0^\circ 28' 48''$ *into decimal degrees and round off your answer to three decimal places. Do the work in the space below.*

3. 0.0988° *in degrees, minutes and seconds, (rounded off to the nearest second), is equal to:*

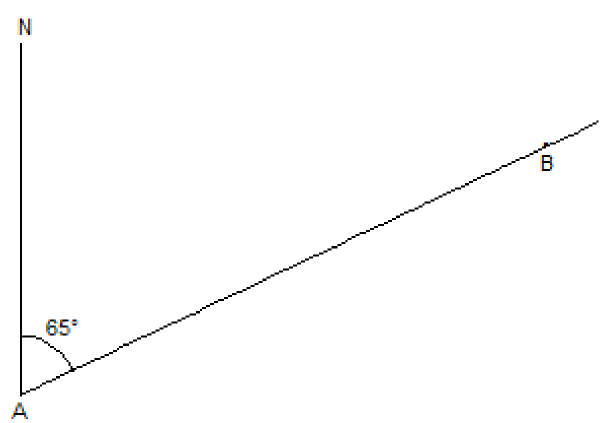
   A) $0^\circ 5' 56''$ *B)* $0^\circ 5' 55''$ *C)* $0^\circ 5' 53''$ *D)* $0^\circ 5' 51''$

4. *Convert* 343.45° *into degrees, minutes and seconds. Do the work in the space below.*

Bearings, (US Army definition)

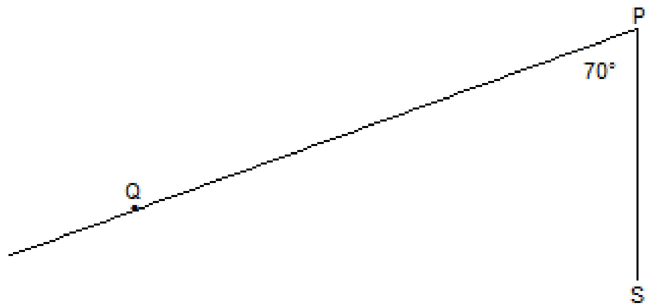
Bearings are used to specify the directions of points of interest relative to some fixed point, with North or south as the reference directions. It is hard to define them but relatively easy to describe how to determine them. Here is how: Say you want to specify directions of points of interest relative to a give point A. Say B is a point of interest and you wish to specify its direction. Standing at A, face either North or Nouth then determine the number of degrees, (which must be less than 90°), you have to turn east or west to face B. It is the condition that you must turn less than 90° which tou use to determine whether to face North or South.

Example 9 *In the figure below, the point of interest is B and its bearings from the given point A have to be determined. To do so, you must stand at A, face north and determine the number of degrees, (they must be less than 90°), you have to turn in order to face B. They turn out to be 65° towards the east. (If you face south, the number of degrees you have to turn to face B is bigger than 90° and that is unacceptable.)*



Face north then turn 65° east

Example 10 *In the figure below, the point of interest is Q and its bearings from the given point P have to be determined. To do so, you must face south and determine the number of degrees, (they must be less than 90°), you must turn in order to face Q. According to the figure, you must turn 70° towards the west.*



Face south then turn 70° west

In Example 9, the bearings of B from A are $N65°E$, (because you face north and turn 65° east to face B). In Example 10, the bearings of Q from P are $S70°W$.

- In general, *the bearings of a point B from a given point A is the* ***acute*** *angle between the ray through A in the direction of North and the ray AB* **or** *the* ***acute*** *angle between the ray through A in the direction of South and the ray AB.*

Example 11 *Consider the figure below in which three points A, B and C are given.*

To determine the bearings of B from A you must stand at A and face south. (If you face north you will have to turn more than 90° to face B.) You must turn 23° *towards the west, (i.e. turn* $90° - 67°$ *towards the west), to face B. Therefore the bearings of B from A are* $S23°W$.

To determine the bearings of C from A you must stand at A and face north. (If you face south you will have to turn more than 90° to face C.) You must turn 52° *towards the east, (i.e. turn* $90° - 38°$ *towards the west), to face C. Therefore the bearings of B from A are* $N52°E$.

What about the bearings of A from B? To determine them, imagine standing at B. You must face north and turn 23° *east. Therefore the bearings of A from B are* $N23°E$.

Following a similar proceedure, one easily shows that the bearings of A from C are $S52°W$.

Practice Problems Set 2, v1

1. The "minutes" hand of a clock turns 360° in 60 minutes, (ignore the sign of the angle).

   (a) How many degrees does it turn in 1 minute?

   (b) How many degrees does it turn between 8:15 am and 8:50 am?

2. Convert $121°15'36''$ into decimal degrees. Write down all the steps to your final answer.

3. Convert 174.857° into degrees, minutes and seconds and round off to the nearest second.

4. The figure shows the positions of four ships A, B, C and D out at sea.

   What is the bearing of:

   (a) B from A?

   (b) C from A?

   (c) A from D?

Practice Problems Set 2, v2

1. The "hour" hand of a clock turns $360°$ in 12 hours, (ignore the sign of the angle).

   (a) How many degrees does it turn in 1 minute?

   (b) How many degrees does it turn between 7:00 am and 9:40 am?

2. Convert $95°35'49''$ into decimal degrees and round off your answer to 3 decimal places. Write down all the steps to your final answer.

3. Convert $36.473°$ into degrees, minutes and seconds and round off to the nearest second.

4. The figure shows the positions of four ships A, B, C and D out at sea.

What is the bearing of:

(a) D from A?

(b) C from A?

(c) A from B?

Practice Problems Set 2, v3

1. The "minutes" hand of a clock turns 360° in 60 minutes, (ignore the sign of the angle).

   (a) How many degrees does it turn in 1 minute?

   (b) How many degrees does it turn between 8:33 am and 9:20 am?

2. Convert $121°48'54''$ into decimal degrees. Write down all the steps to your final answer.

3. Convert 211.658° into degrees, minutes and seconds and round off to the nearest second. Write down all the steps to your final answer.

4. The figure shows the positions of four ships A, B, C and D out at sea.

What is the bearing of:

(a) B from A?

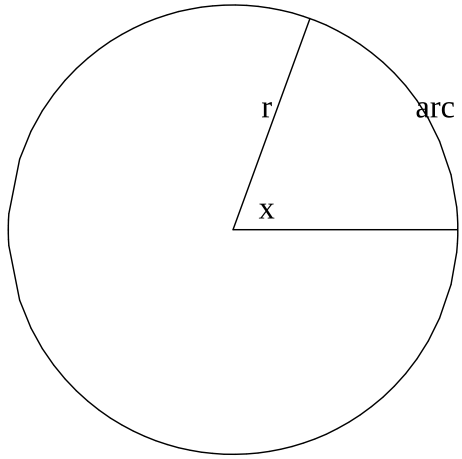
(b) D from A?

(c) A from C?

Length of an arc of a circle subtended by an angle measured in degrees

The figure below shows a circle of radius r inches, (any units may be used for the radius), an angle of x degrees and a segment of the circle that is opposite the given angle. The segment is called **the arc subtended by**

the angle x.



The question we want to answer is: *What is the length of the arc in terms of x?* To answer it, note that an angle of 360° subtends the circumference of the circle which we know to be $2\pi r$ inches long. Therefore:

- An angle of 180°, (which is half of 360°), subtends a semicircle which has length πr.
- An angle of 90°, (which is one quarter of 360°), subtends $\frac{1}{4}$ of a circle which has length $\dfrac{\pi r}{2}$.
- An angle of 1° subtends an arc of length $\dfrac{\pi r}{180}$, (simply divide $2\pi r$ by 360).
- An angle of 2° subtends an arc of length $2 \times \dfrac{\pi r}{180} = \dfrac{2\pi r}{180}$ inches, (simply double $\dfrac{\pi r}{180}$).
- An angle of 35° subtends an arc of length $\dfrac{35\pi r}{180}$ inches.
- In general, an angle of $x°$ subtends an arc of length $x \times \dfrac{\pi r}{180} = \dfrac{\pi r x}{180}$ inches.

We record this finding for use from now on:

In a circle of radius r units, an angle of $x°$ subtends an arc of length $\dfrac{\pi r x}{180}$ units.

Example 12 *In a circle of radius 25 inches, an angle of 92° subtends an arc of length $\dfrac{(25)\,(92)\,\pi}{180}$ inches. This is close to 40.1 inches.*

Example 13 *In a circle of radius 4.9 meters, an angle of 148° subtends an arc of length $\dfrac{(4.9)\,(148)\,\pi}{180}$ meters. This is close to 12.7 maters.*

Example 14 *To calculate the measure of an angle that subtends an arc of length 45 cm in a circle of radius 15 cm:*

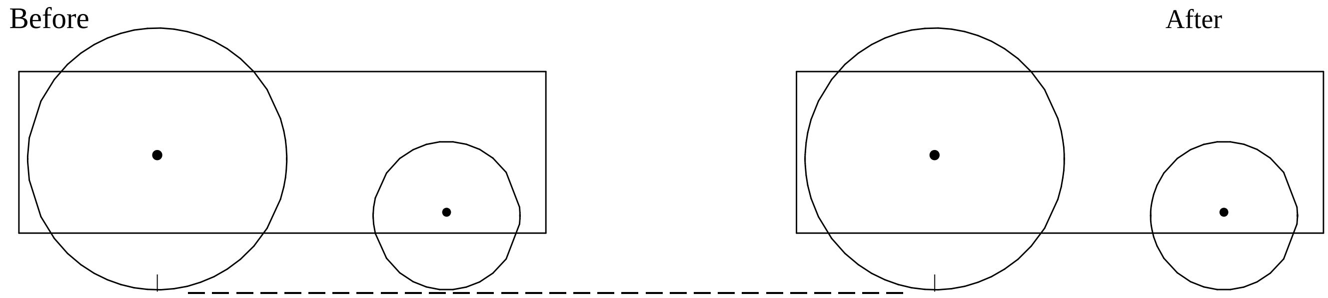
Solution *We are given the arclength and we are required to determine the angle. Let the measure of the angle be x. Using the formular for arclength gives*

$$\frac{\pi \times 15 \times x}{180} = 45$$

Therefore $x = \dfrac{180 \times 45}{\pi \times 15} = 171.9°$, *to 1 decimal place.*

Arc lengths may be used to answer questions about rotating objects like tires. Here is an example:

Example 15 *The figure below shows the main shape of a farm tractor before and after its rear tire has turned 360°. The radius of the tire is 3 feet. The tractor moves a distance equal to the length of the dotted line segment. That length is* $2\pi(3) = 6\pi$ *feet, (the circumference of the tire). To see this, imagine the tire painted red. Then the section of the road it rolls over when it tunrs 360° will be painted red and will have a length equal to the circumference of the tire.*



Since the tractor moves 6π *feet when the tire turns 360°, it follows that:*

- *When the tire tuns* $1°$*, the tractor moves* $\frac{6\pi}{360} = \frac{\pi}{60}$ *feet.*
- *When the tire turns* $x°$*, the tractor moves* $\frac{6\pi x}{360} = \frac{\pi x}{60}$ *feet.*
- *To move one foot, the tire must turn* $\frac{60}{\pi}$ *degrees.*
- *To move one mile, the tire must turn* $\frac{60 \times 5280}{\pi} = \frac{316800}{\pi}$ *degrees. Since there are 360 degrees in one revolution, this translates into* $\frac{880}{\pi}$ *revolutions.*

Exercise 16

1. *Find the length of the arc subtended by an angle of 65° in a circle of radius 7 meters.*

   A) 6.35 *meters*   *B)* 8.73 *meters*   *C)* 7.94 *meters*   *D)* 7.15 *meters*

2. *Find the length of the arc subtended by an angle of 50° in a circle of radius 6 feet.*

   A) 5.24 *feet*   *B)* 4.19 *feet*   *C)* 5.76 *feet*   *D)* 4.72 *feet*

3. *A car wheel has a 16-inch radius. Through what angle (to the nearest tenth of a degree) does the wheel turn when the car rolls forward 2 feet?*

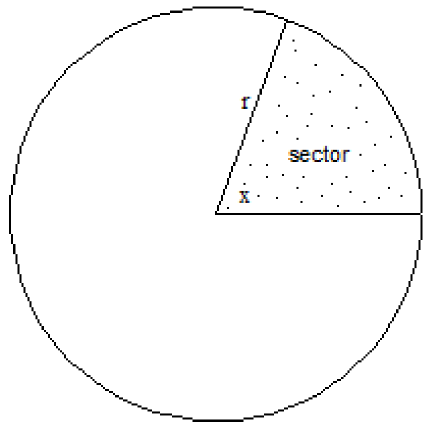
   A) 48.0°   *B)* 58.9°   *C)* 85.9°   *D)* 89.8°

4. *You are given a radius* r *of a circle and an angle* θ*. Calculate the length of the arc subtended by the angle. Round off your answers to 1 decimal place and record them in the given table.*

Radius r	25 *cm*	108 *feet*	3950 *miles*
Angle θ	115.3°	412.4°	3.0288°
Length of arc			

Area of a Sector Subtended by an angle measured in degrees

The figure below shows a circle of radius r inches, an angle of x degrees and **the sector subtended by the angle**.



This time the question to answer is: *What is the area of the sector in terms of x?* We follow the same footsteps that lead us to a formula for the length of an arc: When x is 360°, the sector is the very region enclosed by the circle which we know has area πr^2 square inches. In other words, in a circle of radius r inches, an angle of 360° subtends a sector with area πr^2 square inches. Therefore:

- An angle of 180° subtends a sector with area $\frac{1}{2}\pi r^2$ square inches, (divide πr^2 square inches by 2).
- An angle of 90° subtends a sector with area $\frac{1}{4}\pi r^2$ square inches, (divide πr^2 square inches by 4).
- An angle of 1° subtends a sector with area $\dfrac{\pi r^2}{360}$ square inches, (divide πr^2 square inches by 360).
- An angle of 2° subtends a sector with area $\dfrac{\pi r^2}{360} \times 2 = \dfrac{2\pi r^2}{360}$ square inches, (simply double $\dfrac{\pi r^2}{360}$).
- An angle of 46.4° subtends a sector with area $\dfrac{\pi r^2}{360} \times 46.4 = \dfrac{46.4\pi r^2}{360}$ square inches.
- In general, an angle of $x°$ subtends a sector with area $\dfrac{\pi r^2}{360} \times x = \dfrac{\pi r^2 x}{360}$ square inches.

We record this finding for use from now on:

In a circle of radius r units, an angle of $x°$ subtends a sector with area

$$\frac{\pi r^2 x}{360} \quad \textit{square units.}$$

Example 17 *In a circle of radius 16 cm, the area of the sector, in square centimeters, subtended by an angle of 226° has area*

$$\frac{\pi (16)^2 226}{360}$$

This is approximately equal to 504.9 square centimeters, to 1 decimal place.

Example 18 *In a circle of radius 10 inches, a sector with area 300 square inches is subtended by an angle of x, (in degrees), given by the equation*

$$\frac{\pi (10)^2 x}{360} = 300$$

Solving for x gives

$$x = \frac{300 \times 360}{100\pi} = 343.8° \text{ (rounded off to 1 decimal place).}$$

Exercise 19

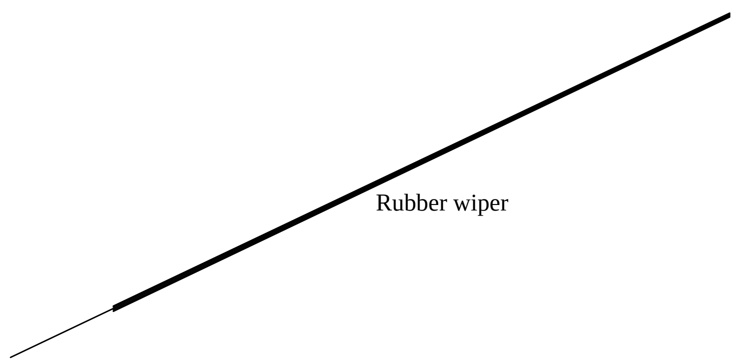
1. *You are given a radius r of a circle and an angle θ. Calculate the area of the sector subtended by the angle. Round off your answers to 1 decimal place and enter them into the table below.*

Radius r	2.5 *cm*	18 *feet*	3950 *miles*
Angle θ	$14.3°$	$24.6°$	$0.981°$
Area of sector			

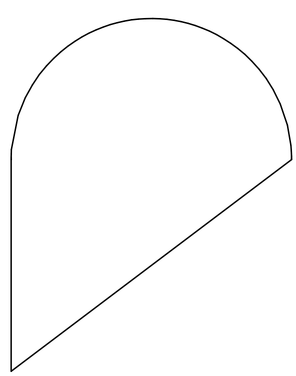
2. *What is the area, to the nearest square centimeter, swept out by a 7 inch minute hand of a clock as it moves between 10.00 am and 10.50 am?*

3. *In a circle of radius r centimeters, an angle of 57° subtends a sector with area 190 sq. centimeters. What is the value of r?*

4. *A car window wiper blade consists of a metal holder 18 inches long to which a 15 inch rubber wiper is attached as shown in the figure below. If it rotates through 110°, how much area does it wipe in one sweep?*



5. *The figure shows a field fenced off by a semicircle and two straight edges. One of the straight edges has length 80 yards and the other has length 100 yards. Calculate the total length of the fencing and the total area that is fenced off.*



6. *There are two slices of pizza. One is $\frac{3}{8}$ of a circular pizza with radius 14 inches and costs \$3.60. The other one is $\frac{2}{3}$ of a circular pizza with radius 10 inches and costs \$3.15. Which of the two gives you more pizza per dollar? You must write down the steps leading to your answer.*

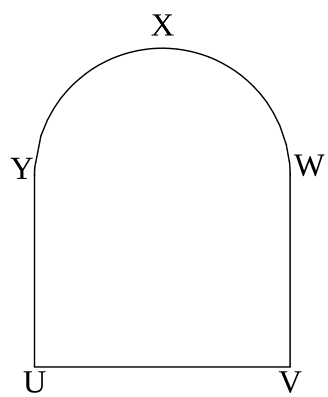
Practice Problems Set 3, v1

1. The minute hand of a clock is 10 cm long. You are required to determine the distance its tip moves between 1:00 pm and 1:48 pm.

   (a) Determine the angle x in degrees, which the minute hand turns between 1:00 pm and 1:48 pm.

   (b) Determine the length of the arc subtended by the angle you have determined, in a circle of radius 10 cm.

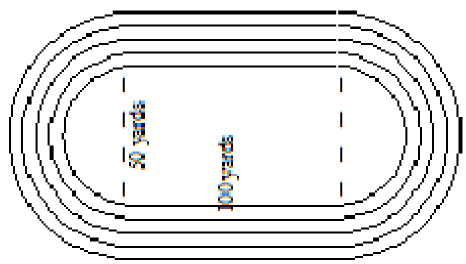
2. The figure shows a Norman window. The section WXY is a semi-circle and the section YUVW is part of a rectangle of width YU equal to 3.8 feet and width UV equal to 4 feet.



   (a) You are required to calculate the total length of decorating ribbon to go around its circumference. Clearly, the section WXY is an arc of a circle. Calculate its length then determine the required length of the decorating ribbon.

(b) What is the total area of the window?

3. A running track consists of an inner rectangle of length 100 yards and width 50 yards, plus semicircular legs at both ends as shown in the figure below. There are 4 lanes and each lane is 1 yard wide.



(a) A runner does one lap in the first lane, (i.e. in the inner-most lane). What is the total distance she covers? (You may assume that she runs along the innermost edge of her lane.)

(b) What distance does a runner in the fourth lane, (i.e. the outer-most lane), when she does one lap in her lane? (You may also assume that she runs along the innermost edge of her lane.) Use this information to determine the head-start a runner in the fourth lane should be given so that she covers the same distance, in one lap, as the runner in the first lane.

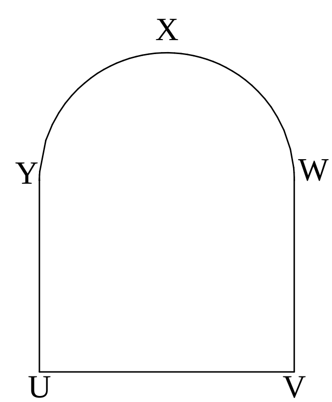
Practice Problems Set 3, v2

1. The minute hand of a clock is 16 cm long. You are required to determine the distance its tip moves between 2:10 pm and 2:48 pm.

   (a) Determine the angle x in degrees, which the minute hand turns between 2:10 pm and 2:48 pm.

   (b) Determine the length of the arc subtended by the angle you have determined, in a circle of radius 16 cm.

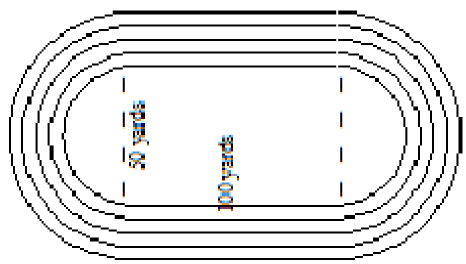
2. The figure shows a Norman window. The section WXY is a semi-circle and the section YUVW is part of a rectangle of width YU equal to 4 feet and width UV equal to 4.6 feet.



   (a) You are required to calculate the total length of decorating ribbon to go around its circumference. Clearly, the section WXY is an arc of a circle. Calculate its length then determine the required length of the decorating ribbon.

(b) What is the total area of the window?

3. A running track consists of an inner rectangle of length 100 yards and width 50 yards, plus semicircular legs at both ends as shown in the figure below. There are 4 lanes and each lane is 1 yard wide.

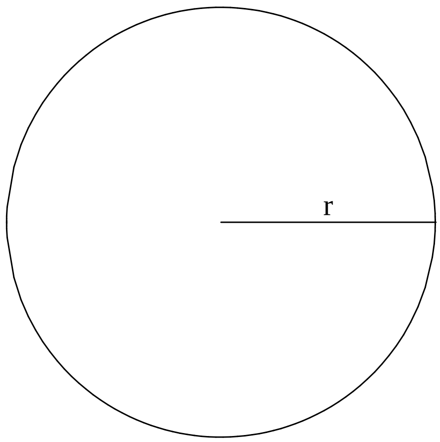


(a) A runner does one lap in the first lane. What is the total distance she covers? (You may assume that she runs along the innermost edge of her lane.)

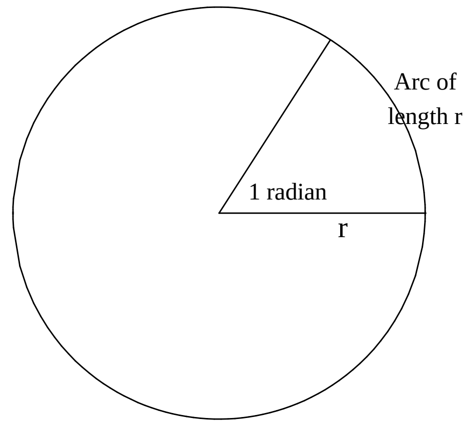
(b) What distance does a runner in the third lane cover when she does one lap in her lane? (You may also assume that she runs along the innermost edge of her lane.) Use this information to determine the head-start a runner in the third lane should be given so that she covers the same distance, in one lap, as the runner in the first lane.

Another unit of angle measure

There are a number of formulas that are much simpler when angles are measured in a completely different unit called a **radian**. To draw one radian, take any circle with a big enough radius r, (an example is given below), and measure off an arc of length r. The angle subtended by the arc is one radian.

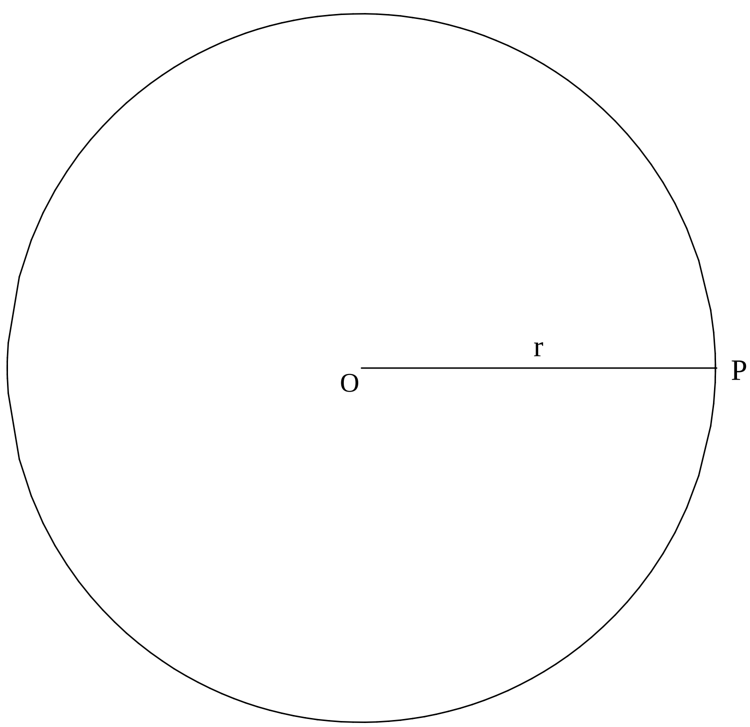


A circle of radius r



An angle of 1 radian

Exercise 20 *You are required to draw angles of 3 radians, 2.5 radians and* -1.5 *radians on the figure below. They do not have to be accurate, but they should be reasonable. One way to do this is to take a thin strip of paper that has length r and wrap it around the circle, (starting at* P*). That will give you an angle of one radian. To get an angle of two radians you may use a strip that is twice as long or simply use the first srtip twice. The required angles should be obtained in the same way*



To get an idea of how big a radian is compared to a degree, note that the circumference of a circle with radius r is $2\pi r$, and it subtends an angle of $360°$.

In other words, an arc of length $2\pi r$ is subtended by an angle of $360°$.

It follows that an arc of length 1 unit is subtended by an angle of $\frac{360}{2\pi r}$ degrees.

Therefore an arc of length r is subtended by an angle of $\frac{360}{2\pi r} \times r = \frac{360}{2\pi} = \frac{180}{\pi}$ degrees.

Since $\frac{180}{\pi} = 57.3$ to one decimal place, we conclude that one radian is approximately equal to 57.3 degrees. In some cases we may want to use the exact value which is

$$\boxed{1 \text{ radian } = \frac{360}{2\pi} = \frac{180}{\pi} \text{ degrees}}$$

It follows that x radians are equal to $\frac{180x}{\pi}$ degrees. We record this for later use:

$$x \text{ radians equal } \frac{180x}{\pi} \text{ degrees} \tag{1}$$

Example 21

$$2 \textit{ radians} = 2 \times \tfrac{180}{\pi} \textit{ degrees which is approximately } 114.59°$$

$$2.45 \textit{ radians} = 2.45 \times \tfrac{180}{\pi} \textit{ degrees which is approximately } 140.37°$$

$$-3.5 \textit{ radians} = -\frac{180 \times 3.5}{\pi} \textit{ degrees which is approximately } 200.54°$$

We can also easily change from degrees into radians, by simply working backwards. More precisely, since $\frac{180}{\pi}$ degrees are equal to one radian, it follows that 1 degree equals $\frac{\pi}{180}$ radians. This is worth recording:

$$\boxed{1 \text{ degree } = \frac{\pi}{180} \text{ radians.}}$$

In general, y degrees equal $\frac{\pi y}{180}$ radians. We record this for later use:

$$\boxed{y \text{ degrees } = \frac{\pi y}{180} \text{ radians.}}$$

Example 22

$$2 \textit{ degrees} = 2 \times \frac{\pi}{180} \textit{ radians which is approximately } 0.035 \textit{ radians}$$

$$108.7 \textit{ degrees} = 108.7 \times \frac{\pi}{180} \textit{ radians which is approximately } 1.897 \textit{ radians}$$

$$-443 \textit{ degrees} = -\frac{443\pi}{180} \textit{ radians which is approximately } -7.731 \textit{ radians.}$$

The angles $30°$, $45°$, $60°$, and $90°$ are particularly common in trigonometric problems. For that reason, it is suggested that you memorize their corresponding measures in radians. They are

$$30° = \frac{30\pi}{180} = \frac{\pi}{6} \text{ radians} \qquad 45° = \frac{45\pi}{180} = \frac{\pi}{4} \text{ radians}$$

$$60° = \frac{60\pi}{180} = \frac{\pi}{3} \text{ radians} \qquad 90° = \frac{90\pi}{180} = \frac{\pi}{2} \text{ radians}$$

Their multiples can be determined by multiplying the corresponding radian measure with the appropriate factor. For example, since 120° is twice 60°,

$$120° = 2 \times \frac{\pi}{3} \text{ radians } = \frac{2\pi}{3} \text{ radians}$$

Since 135° is three times 45°, and 150° is five times 30°,

$$135° = 3 \times \frac{\pi}{4} \text{ radians } = \frac{3\pi}{4} \text{ radians and } 150° = 5 \times \frac{\pi}{6} = \frac{5\pi}{6} \text{ radians}$$

Exercise 23

1. *Convert an angle of* $\frac{9\pi}{6}$ *radians into degrees*

   A) 540° *B)* 270° *C)* 120π° *D)* 160°

2. *Convert an angle of* 105° *into radians and give an answer that is a multiple of* π.

   A) $\frac{1}{2}\pi$ *radians* *B)* $\frac{6\pi}{11}$ *radians* *C)* $\frac{8\pi}{13}$ *radians* *D)* $\frac{7\pi}{12}$ *radians*

3. *Convert an angle of* $\frac{7\pi}{2}$ *radians into degrees* *(SLO 1d)*

   A) 51π° *B)* 630° *C)* 1260° *D)* 154°

4. *Convert an angle of* 315° *into radians and give an answer that is a multiple of* π.

   A) $\frac{7\pi}{3}$ *radians* *B)* $\frac{7\pi}{4}$ *radians* *C)* $\frac{7\pi}{5}$ *radians* *D)* $\frac{7\pi}{8}$ *radians*

5. *Convert* $\frac{25}{9}\pi$ *radians into degrees.*

   A) 1000π° *B)* 9° *C)* 500° *D)* 250°

6. *Convert each angle, (given in degrees), into radians. When necessary, round off your answer to 2 decimal places.*

Given angle in degrees	*210°*	*300°*	−330°	*225°*	*109°*	*255.4°*	−447.88°
Converted angle in radians							

7. *Convert* −480° *into radians and express your answer as a multiple of* π.

   A) $-\frac{8\pi}{3}$ *radians* *B)* $-\frac{7}{3}\pi$ *radians* *C)* $-\frac{7\pi}{2}$ *radians* *D)* $-\frac{9\pi}{4}$ *radians*

8. *Convert each angle, (given in radians), into degrees. When necessary, round off your answer to 1 decimal place.*

Given angle in radians	2.3 *radians*	$\frac{3\pi}{7}$ *radians*	−5.53 *radians*	$\frac{\pi}{15}$ *radians*	$\frac{11\pi}{6}$ *radians*	$-\frac{18\pi}{15}$ *radians*
Converted angle in degrees						

Length of an arc subtended by an angle measured in radians

One advantage radians have over degrees is that we calculate arc-lengths much more easily when angels are measured in radians. To see how simple, note that in a circle of radius r units, an angle of 1 radian subtends an arc of length r units. It follows that:

an angle of 2 radian subtends an arc of length $2r$ units,

an angle of 4.8 radian subtends an arc of length $4.8r$ units,

an angle of x radians subtends an arc of length rx units.

We record these observations for later use:

In a circle of radius r units, an angle of x radians subtends an arc of length rx units.

Example 24 *In a circle of radius 28 cm, an angle of 2.5 radians subtends an arc of length* $28 \times 2.5 = 70.0$ *cm.*

Example 25 *In a circle of radius 3 feet, an arc of length 6.9 ft. is subtended by an angle of* $\dfrac{6.9}{3} = 2.3$ *radians.*

Example 26 *If an arc of length 66 meters is subtended by an angle of 4.8 radians then the circle must have radius* $\dfrac{66}{4.8} = 13.75$ *meters.*

Exercise 27

1. *Find the length of the arc subtended by an angle of the given measure in a circle with the given radius. Round off your answers to 1 decimal place and record them in the given table.*

Radius r	4.9 *m*	1.6 *yards*	3000 *Kilometers*
Angle θ	*3.77 radians*	4.8 *radians*	0.041 *radians*
Length of arc			

Area of a sector subtendd by an angle measured in radians

The area of a sector subtended by an angle is also relatively easier to calculate when the angle is given in radians. Recall that the area enclosed by a circle of radius r is πr^2 square units and it is the sector subtended by an angle of 2π radians. In other words, an angle of 2π radians subtends a sector with area πr^2. Therefore an angle of 1 radian subtends a sector with area $\dfrac{\pi r^2}{2\pi} = \dfrac{r^2}{2}$ square units. This implies that:

an angle of 2 radian subtends a sector with area $\dfrac{2r^2}{2}$ square units

an angle of 5.1 radians subtends a sector with area $\dfrac{5.1r^2}{2}$ square units

an angle of x radians subtends a sector with area $\dfrac{xr^2}{2} = \frac{1}{2}r^2x$ square units

We also record this for later use:

In a circle of radius r units, an angle of x radians subtends a sector with area $\frac{1}{2}r^2x$ square units.

Exercise

1. Find the radian measure of the angle subtended by an arc of the given length, in a circle with the stated radius. In each case, draw a diagram.

Radius of circle	Length of arc	Radian measure of angle
28 centimeters	12 centimeters	
4 meter	19 meters	
300 kilometers	250 kilometers	

2. Find the area of the segment subtended by an angle of the given measure, in radians, in a circle with the given radius. In each case, draw a diagram.

Radius of circle	Measure of angle	Area of segment
5 inches	5.4 radians	
5 centimeters	0.79 radians	
4000 kilometers	0.082 radians	

Angular Speed of a Rotating Object

Angular speeds arise where there are rotating objects like car tires, engine parts, computer discs, etc. When you drive a car, a gauge on the dashboard, (called a tachometer), continuously displays how fast your engine, (actually the crankshaft in your engine), is rotating. The number it displays at a particular time is called the angular speed of the engine at that instant, in *revolutions per minute*, (abbreviated to rpm). For example, an angular speed of 3000 rpm means that the engine rotates 3000 times in one minute. Since there isn't much space on the dashboard, 3000 rpm is actually displayed as 3 and you are instructed to multiply it by 1000, (usually indicated by a symbol $\times 1000$ in the center of the tachometer).

For objects like the seconds hand of a clock, the minute hands of a clock, etc, that rotate relatively slowly, the angular speed is often given in degrees or radians per unit time. For example, the angular speed of the seconds hand of a clock may be given as $360°$ per minute, or $6°$ per second, (divide 360 by 60), or $21600°$ per hour, (multiply 360 by 60).

In general, to determine the angular speed of a rotating object per unit time do the following:

(i) Fix a time interval

(ii) Record the amount of turning, (in degrees or radians or revolutions), in that time period,

(iii) Divide the amount of turning by the length of the time interval.

Example 28 *Wanda noticed that when the ceiling fan in her room is set to operate at "very low" speed, it makes a full rotation in 5 seconds.*

This means that a point on the fan sweeps out an angle of 360° in 5 seconds. Therefore its angular speed is

$$\omega = \frac{360}{5} = 72 \textit{ degrees per second}$$

If, instead, we measure angles in radians, then it sweeps out 2π radians in 5 seconds therefore its angular speed is

$$\omega = \frac{2\pi}{5} \textit{ radians per second}$$

Exercise 29

1. *Express an angular speed of 12 revolutions per second into radians per second.*

   A) 12 *radians per sec.* *B)* 12π *radians per sec.* *C)* 24 *radians per sec.* *D)* 24π *radians per sec.*

2. *What is the angular speed of the hour hand of a clock in:*

   (a) *Degrees per hr.*
   (b) *Degrees per min.*
   (c) *Radians per hr.*
   (d) *Revolutions per min.*

Linear Speed of a Rotating Object

The tires of a vehicle must rotate in order for the vehicle to move. When they rotate clockwise, it moves forward. It moves back when they rotate counter-clockwise. For a specific example, consider the tractor in Example 15. We observed that when the rear tire turns 360°, the tractor moves 6π feet. Say the tractor is cruising on a road and it is observed that the rear tire makes 2 full revolutions per second. This means that its angular speed is 720° per second. We noted that the tractor moves $\frac{\pi}{60}$ feet when the tire turns 1°. Therefore it moves

$$\frac{\pi}{60} \times 720 = 12\pi \text{ feet per second}$$

The figure 12π feet per second is called the **linear speed** of points on the rim of the tire. We may give it in other units. For example, since there are 3600 seconds in 1 hour, the tractor moves

$$12\pi \times 3600 = 43200\pi \text{ feet per hour}$$

This is the linear speed of points on the rim of the tire in feet per hour. We may convert it into miles per hour by dividing it by 5280, (because there are 5280 feet in 1 mile). Therefore the tractor moves

$$\frac{43200\pi}{5280} \text{ miles per hour}$$

This is close to 25.7 miles per hour. It is the linear speed of a point on the rim of the tire in miles per hour.

In general, suppose a circular object of radius r units is rotating at the rate of y degrees per unit time. The units for r may be feet, inches, meters, etc and the unit of time may be a second or a minute or an hour. Imagine that the object is a tire on a vehicle. Then in unit time, the vehicle moves a distance equal to the length of the arc subtended by an angle of y degrees in a circle of radius r. That length is $\frac{\pi r y}{180}$ and it is called the linear speed of points on the edge of the rotating object. Note that y is the angular speed of the rotating object. Therefore the linear speed of a point on the edge of the rotating object is equal to

$$\left(\frac{\pi}{180}\right) \times (\text{angular speed of rotating object}) \times (\text{radius of rotating object})$$

Its units are determined by the units of the radius and time interval. For example, if r is in feet and y is in seconds then the units of linear speed will be feet per second. We record this for later use:

- *The linear speed of points on the edge of a rotating object that has radius r and is rotating at the rate of y degrees per unit time is*

$$\frac{\pi y r}{180}$$

Its units are derived from the unit of time and the units for the the length of the radius.

If angles are measured in radians them the expression for linear speed is even simpler. Say a circular object with radius r is rotating at the rate of ω radians per unit time. Imagine that it is a tire on a vehicle. Then in unit time, the vehicle moves a distance equal to the length of the arc subtended by an angle of ω radians in a circle of radius r. That length is $r\omega$ units and it is the linear speed of points on the edge of the rotating object. This is also worth recording:

- *The linear speed of points on the edge of a rotating object that has radius r and is rotating at the rate of ω radians per unit time is*

$$r\omega$$

Its units are derived from the unit of time and the units for the the length of the radius.

Example 30 *A Farris wheel, pictured below, in an amusement park has radius 25 feet and it makes a full rotation every 1.6 minutes.*

What is the linear speed of a point on the rim of the wheel in (i) feet per minute, (ii) feet per second, (iii) yards per minute, (iv) miles per hour?

Solution: *Since the wheel turns 2π radians in 1.6 minutes, its angular speed $\frac{2\pi}{1.6}$ radians per minute. We are given that its radius is 25 feet. Therefore the linear speed of points on the rim of the wheel is:*

(i) $25\left(\frac{2\pi}{1.6}\right)$ *feet per minute. This is approximately equal to 98 feet.*

(ii) $25\left(\frac{2\pi}{1.6}\right) \div 60$ *feet per second because there are 60 seconds in a minute. To 1 decimal place, it is 1.6 feet per second.*

(iii) $25\left(\frac{2\pi}{1.6}\right) \div 3$ *yards per minute because there are 3 feet in one yard. To 1 decimal place, it is 32.7 yards per minute.*

(iv) $25\left(\frac{2\pi}{1.6}\right) \times 60 \div 5280$ *miles per hour because there are 60 minutes in one hour and 5280 feet in one mile. To 1 decimal place, this is 1.8 miles per hour.*

Exercise 31

1. *A wheel with radius 50 cm is turning at the rate of* $\frac{2\pi}{3}$ *radians per second. What is the linear speed of a point on the outer edge of the wheel?*

   A) $\dfrac{100\pi}{3}$ *cm per sec.* *B)* $\dfrac{125\pi}{3}$ *cm per sec.* *C)* $\dfrac{150\pi}{3}$ *cm per sec.* *D)* $\dfrac{155\pi}{3}$ *cm per sec.*

2. *A gear with a radius of 2 centimeters is turning at* $\dfrac{\pi}{11}$ *radians per second. What is the linear speed at a point on the outer edge of the gear?*

   A) 22π *centimeters per second* *B)* $\dfrac{11\pi}{2}$ *centimeters per second*

   C) $\dfrac{2\pi}{11}$ *centimeters per second* *D)* $\dfrac{\pi}{22}$ *centimeters per second*

3. *A pick-up truck is fitted with new tires which have a diameter of 41 inches. The tires are rotating at 370 revolutions per minute.*

   (a) What is the angular speed of the tires in radians per hour?

   A) 740π *rad. per hr.* *B)* 1110π *rad. per hr.* *C)* 4440π *rad. per hr.* *D)* 44400π *rad. per hr.*

   (b) How fast, in miles per hour, is the truck moving? (1 mile equals 63360 inches.)

   A) 0.8 *mph* *B)* 23 *mph* *C)* 45 *mph* *D)* 50 *mph*

4. *Each tire of a truck has radius 1.8 feet. Assume that they make 10 complete revolutions per second. Calculate the angular speed of the tires in degrees per second. Also, calculate the linear speed of a point on the outer edge of the tire in (i) feet per second, (ii) miles per hour.*

5. *Each tire of a certain car has radius 1.2 feet. At what angular speed, in revolutions per second, are the tires turning when it is moving at a constant speed of 70 miles per hour? (The angular speed of the tires may be different from the angular speed of the engine because of the gears.)*

6. *An object moves along a circle of radius 10 meters and sweeps out an angle of* $\frac{9\pi}{2}$ *radians per minute. Calculate its linear speed in, (i) meters per minute, (ii) kilometers per hour.*

7. *An object moves along a circle of radius 9 feet and sweeps out an angle of* 60° *per second. Calculate its linear speed in, (i) feet per minute, (ii) miles per hour.*

Practice Problems Set 4, v1

1. Convert $\frac{7\pi}{8}$ radians into degrees. Give the exact answer.

2. Convert 390° into radians and give your answer as a multiple of π

3. Calculate the length of the arc subtended by an angle of 3.8 radians in a circle of radius 4 inches

4. Calculate the area of the sector subtended by an angle of 3.5 radians in a circle of radius 6 inches

5. The radius of the earth is about 3960 miles. It turns a full revolution in 24 hours.

   (a) Calculate the angular speed of the earth in radians per hour.

   (b) Determine the linear speed of a point on the equator in miles per hour

Practice Problems Set 4, v2

1. Convert $\frac{6\pi}{5}$ radians into degrees. Give the exact answer.

2. Convert 540° into radians and give your answer as a multiple of π

3. Calculate the area of the sector subtended by an angle of 5.3 radians in a circle of radius 7 inches

4. Calculate the area of the sector subtended by an angle of 2.6 radians in a circle of radius 5 inches

5. The radius of the earth is about 3960 miles and there are 5280 feet in a mile. It turns a full revolution in 24 hours.

   (a) Calculate the angular speed of the earth in radians per minute.

   (b) Determine the linear speed of a point on the equator in feet per minute

Practice Problems Set 4, v3

1. Convert $\frac{7\pi}{5}$ radians into degrees. Give the exact answer.

2. Convert 480° into radians and give your answer as a multiple of π

3. Calculate thelength of the arc subtended by an angle of 3.8 radians in a circle of radius 4 inches

4. Calculate the area of the sector subtended by an angle of 2.5 radians in a circle of radius 4 inches

5. The radius of the earth is about 3960 miles and there are 63360 inches in a mile. It turns a full revolution in 24 hours.

   (a) Calculate the angular speed of the earth in radians per second.

   (b) Determine the linear speed of a point on the equator in inches per second